Box Partitions

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Introduction

Recall that the Riemann integral of a function $f : \mathbb{R}^n \to \mathbb{R}^m$ is defined by partitioning the region of integration into boxes of size $\Delta V = \Delta x_1 \Delta x_2 \cdots \Delta x_n$ and letting Δx_i approach 0 for each i. The value $f(\mathbf{x}_\beta)\,\Delta V$ is summed over all boxes β, where $\mathbf{x}_\beta$ is any point inside β. The integral is the limit of this sum as the box size goes to 0.

Box partitions are a way of generalizing this to functions with any Hausdorff space as their domain, and a complete normed vector space (Banach space) as their codomain. A box partition is a net (a generalization of sequence) of finite partitions whose elements "shrink". Instead of defining the integral of a function, we will define the mean. There is a relatively simple condition that can be put on a box partition of a compact Hausdorff space to guarantee that the mean of any continuous function exists. Also, it turns out that every compact regular Hausdorff space has a box partition, assuming the axiom of choice.

Notation and Background

Here is the notation we will use in this poster:

- X is a Hausdorff space,
- A is a directed set,
- B is a Banach space over $\mathbb{R}$ or $\mathbb{C}$, and
- f is a continuous function from X to B.

Background:

- A *Banach space* is a vector space with a norm $\|\cdot\|$ that is complete (any Cauchy sequence converges).
- A *directed set* is a set A with a relation "$\geq$" that is reflexive, transitive, and such that every pair of elements has an upper bound.
- Given a family (A_i) of directed sets, their Cartesian product $\prod A_i$ is also a directed set, where $(a_i) \geq (b_i)$ if and only if $a_i \geq b_i$ *for all* i in the index set. (The relation "$\geq$" does not have to be a total order.)
- A *net* is an indexed family $(x_a)_{a\in A}$, where A is a directed set. When the x_a are in a Hausdorff space X, we will write $\lim_{a\to\infty} x_a$ for the value to which (x_a) converges (if it exists).
- A topological space X is *regular* if for any point $x \in X$ and any closed set C not containing x, there exist disjoint neighborhoods of x and C.
- A topological space X is *compact* if every open cover of X has a finite subcover.

Definitions

Definition

A *box partition* of X is a net $(P_a)_{a\in A}$ of finite partitions of X that satisfies the *local shrink condition*:
Let U be a nonempty open set, and let $x \in U$. Then, there is some neighborhood V of x such that for all sufficiently large a, all elements of P_a that intersect V are contained in U.

Note: For a box partition (P_a), we will call the elements of P_a "boxes".

Definition

A *selector* is a net $(X_a)_{a\in A}$ of finite subsets of X such that for any $x \in X$, there exists a net $(x_a)_{a\in A}$ that converges to x, where $x_a \in X_a$.

Definition

Let (P_a) be a box partition. A selector (X_a) is called a *selector for* (P_a) if for all $a \in A$ and all $\beta \in P_a$, the set $X_a \cap \beta$ contains exactly one element. This means that a selector for a box partition chooses one element from each box.

Definition

Let (X_a) be a selector in X. The *mean* of a function $f : X \to B$ using the selector (X_a) is defined as the following limit, if it exists:

$$\operatorname{mean} f = \lim_{a\to\infty} \frac{1}{|X_a|} \sum_{x\in X_a} f(x).$$

Definition

Let (X_a) be a selector. Define the *relative measure with respect to* (X_a) of two subspaces R and S to be

$$|R : S| = \lim_{a\to\infty} \frac{|X_a \cap R|}{|X_a \cap S|},$$

if the limit exists.

Definition

If (X_a) is a selector for a box partition (P_a), then we say that (P_a) is *uniform with respect to* (X_a) if

$$\lim_{a\to\infty} \max_{\beta,\gamma\in P_a} |\beta : \gamma| = 1,$$

where $|\beta : \gamma|$ denotes the relative measure with respect to (X_a).

Examples

Here are some examples of box partitions and selectors:

Example

The unit interval $[0,1]$ with the standard topology has a box partition $(P_n)_{n\in\mathbb{N}}$, where

$$P_n = \left\{ \left[0, \tfrac{1}{n}\right), \left[\tfrac{1}{n}, \tfrac{2}{n}\right), \ldots, \left[\tfrac{n-1}{n}, 1\right] \right\}.$$

Similarly, $[0,1]$ has a selector $(X_n)_{n\in\mathbb{N}}$ where

$$X_a = \left\{0, \tfrac{1}{n}, \tfrac{2}{n}, \ldots, \tfrac{n-1}{n}\right\},$$

and it is a selector for (P_n). If a function $f : [0,1] \to \mathbb{R}^m$ is Riemann integrable, then its mean is equal to its integral.

Example

Any finite Hausdorff space X has a "discrete" box partition $(P_a)_{a\in\{0\}}$, where P_0 is the partition of X into one-element sets.

We can also construct selectors and box partitions from other selectors and box partitions:

Example

If (P_a) is a box partition and $Y \subseteq X$, then there is a *subspace box partition* (P'_a) on Y defined by $P'_a = \{\beta \cap Y \mid \beta \in P_a\}$.

Example

If $(X^{(i)})_{i\in I}$ is a family of Hausdorff spaces with box partitions $(P_a^{(i)})$, then (Q_a) defined by $Q_a = \bigcup P_a^{(i)}$ is a box partition (the *disjoint union box partition*) for $\bigsqcup X^{(i)}$, and (R_a) defined by

$$R_a = \left\{ \prod_{i\in I} \beta^{(i)} \;\middle|\; \beta^{(i)} \in P_a^{(i)} \right\}$$

is a box partition (the *product box partition*) for $\prod X^{(i)}$. The disjoint union selector and product selector of a family of selectors can be defined similarly.
Taking the product of the selector in the first example with itself k times gives a selector for $[0,1]^k$, and the mean of a Riemann integrable function is its integral.

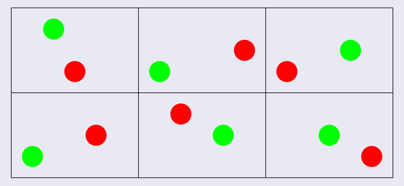
Existence of a Box Partition

Let X be a compact regular Hausdorff space. How can we prove existence of a box partition?

- For each $x \in X$, let A_x be the directed set of all open sets containing x, where $U \geq V$ means $U \subseteq V$. Two elements $U, V \in A_x$ have an "upper bound": their intersection $U \cap V$.
- Take the product $A = \prod_{x\in X} A_x$. This is the directed set we will use as the index set of the box partition.
- For each $a = (U_x) \in A$, let $\mathcal{C}_a$ be the open cover $\{U_x \mid x \in X\}$ of X.
- Since X is compact, there exists a finite subcover $\mathcal{C}'_a = \{U_x \mid x \in Y_a\}$ of $\mathcal{C}_a$, where $Y_a \subseteq X$.
- Given the open cover $\mathcal{C}'_a$ of X, we can "remove the overlap" of the sets in $\mathcal{C}'_a$ to get a refinement (whose elements are not necessarily open) that is a finite partition
$$P_a = \{\beta_{a,x} \mid a \in A, x \in Y\}$$
of X.
- To remove the overlap, let $(y_n)_{n<N}$ be a well-ordering of Y, where N is an ordinal number. (This uses the axiom of choice.) Recursively define $\beta_{a,x}$ for each $x \in Y$ by
$$\beta_{a,y_n} = U_{y_n} \setminus \left(\bigcup_{k<n} \beta_{a,y_k} \right).$$
- Then we get a box partition $(P_a)_{a\in A}$ of X, as long as the local shrink condition is satisfied.
- To prove the local shrink condition for (P_a), let U be a neighborhood of a point $x_0 \in X$, and we will find an index $a_0 \in A$ such that for all $a \geq a_0$, the box in P_a containing x_0 is contained in U. By Theorem 12, this is equivalent to the local shrink condition. (Here is where we use regularity of X.)
- For a point $x \in X$, define
$$U_x = \begin{cases} U & \text{if } x \in U, \\ X \setminus \{x_0\} & \text{if } x \notin U. \end{cases}$$
Let $a_0 = (U_x)_{x\in X}$.
- Let $a = (U'_x) \geq (U_x) = a_0$. Let $x \in Y_a$ such that $\beta_{a,x}$ is the box in P_a containing x_0. If $x \notin U$, then $x_0 \in \beta_{a,x} \subseteq U_x = X \setminus \{x_0\}$, which is a contradiction. So $x \in U$, which means $\beta_{a,x} \subseteq U$.
- Therefore, (P_a) satisfies the local shrink condition, so it is a box partition.

Two Selectors for a Box Partition

Let X be compact, and let (P_a) be a box partition of X.



The red and green points are from two different selectors (X_a) and (Y_a) for the box partition (P_a). Let $f : X \to B$ be continuous, and let $\epsilon > 0$. By Theorem 7, the diameter of $f(\beta)$ is less than ϵ for all boxes $\beta \in P_a$ for sufficiently large $a \in A$. So for all sufficiently large a, all pairs of selector points x and y that are in the same box in P_a satisfy $\|f(x) - f(y)\| < \epsilon$, which means that the *average* distance between the two selectors (over all boxes in P_a) is also less than ϵ. This average distance approaches (as $a \to \infty$) the difference between the mean of f using (X_a) and the mean of f using (Y_a). Therefore, the means are the same. This proves Theorem 8.

Theorems

1. Given a selector (X_a), the set of all continuous functions $f : X \to B$ for which the mean exists (when using (X_a) as the selector) is a vector space (under pointwise addition and scalar multiplication). On that vector space, the function that maps f to mean f is linear.
2. Let B and C be Banach spaces, and let $L : B \to C$ be a continuous linear map. For any selector (X_a), and function $f : X \to B$ such that mean f exists, we have
$$\operatorname{mean} L \circ f = L(\operatorname{mean} f).$$
3. Suppose that the Banach space B is a Banach algebra, with a product operator "$\cdot$". Let X and Y be Hausdorff spaces with selectors (X_a) and (Y_a) respectively, and let $f : X \to B$ and $g : Y \to B$. Define a function $h : X \times Y \to B$ by $h(x,y) = f(x)\cdot g(y)$. If $\operatorname{mean}_X f$ and $\operatorname{mean}_Y g$ exist, then
$$\operatorname{mean}_{X\times Y} h = (\operatorname{mean}_X f)\cdot(\operatorname{mean}_Y g),$$
using the product selector of (X_a) and (Y_a).
4. Given a box partition, there exists a selector for that box partition, assuming the axiom of choice.
5. Let X be finite, and let (P_a) be a box partition. For all sufficiently large a, P_a is the partition of X into one-element sets.
6. If X is finite, the mean of a function $f : X \to B$ exists when using any selector in X, and it always agrees with the usual definition of mean.
7. Let (P_a) be a box partition of X, let $S \subseteq X$ be compact, and let $f : X \to B$ be continuous. For all $\epsilon > 0$, there exists an $a_0 \in A$ such that for all $a \geq a_0$, any box $\beta \in P_a$, and any $x, y \in \beta \cap S$, we have $\|f(x) - f(y)\| < \epsilon$. (This is similar to the theorem that on a compact subspace of a metric space, every continuous function is uniformly continuous.)
8. Let (P_a) be a box partition of a compact Hausdorff space X. Then, the mean of a function $f : X \to B$ is the same using any selector (X_a) for the box partition (P_a). This allows us to define the mean of f if we have a box partition but haven't chosen a selector for it (assuming the axiom of choice).
9. Let X be the disjoint union of two compact Hausdorff spaces R and S. Let (P_a) be a box partition for X, and let $f : X \to B$ be continuous. If $\operatorname{mean}_R f|_R$ and $\operatorname{mean}_S f|_S$ exist when using the subspace box partitions, then
$$\operatorname{mean} f = |R : X| \cdot \operatorname{mean}_R f|_R + |S : X| \cdot \operatorname{mean}_S f|_S.$$
10. If (P_a) is uniform with respect to one selector (X_a) for (P_a), then it is uniform with respect to *every* selector for (P_a), and we say that (P_a) is a *uniform box partition*.
11. When using a uniform box partition of a compact Hausdorff space, the mean of a continuous function always exists.
12. If X is a compact regular Hausdorff space, then the local shrink condition on (P_a) is equivalent to the condition that for any neighborhood U of a point $x \in X$, the box in P_a containing x is contained in U for all sufficiently large $a \in A$.
13. Every compact regular Hausdorff space has a box partition, assuming the axiom of choice.

Conjectures

1. Given a selector (X_a), there always exists a box partition (P_a) such that (X_a) is a selector for (P_a).
2. Every compact regular Hausdorff space has a *uniform* box partition.
3. Let $X = [0,1]$ and (X_a) be the selector from the first example. Then, there exist two sets R and S such that the limit defining $|R : S|$ is bounded but oscillates forever.
4. For a uniform box partition of a compact Hausdorff space, Theorem 11 says that the mean of a continuous function always exists. Is this true for any subspace box partition of a uniform box partition?
5. The regularity condition and/or the compactness condition can be removed from Theorem 13.