Period Doubling Cascades from Models and from Data

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Introduction

We investigate the dissimilarity between the orbits of the logistic map and the orbits of a Gaussian process fit regression through sample points corresponding to that logistic map while varying the parameter r, the bifurcation parameter. Such analysis should give us insight on the asymptotic dynamics in the orbit diagrams of real-world models.

Gaussian Process Reconstruction

Model we want to reconstruct:
We focused on the logistic map, a map that is often used to model populations. The logistic map is the recursive relation:

$$x_{n+1} = f(x_n) = rx_n(1 - x_n)$$

where $r \in [0, 4]$ is the bifurcation parameter, $x_0 \in [0, 1]$ is a dimensionless initial population value, and x_n is the population in the nth generation [2].

Method of reconstruction:
We use Gaussian Process (GP) regression to reconstruct the logistic map from N sample points of the logistic map. See Figure 1, which depicts a GP regression off samples points of the logistic map.

- We use the squared exponential kernel.
- The kernel parameters (length-scale and variance) are optimized automatically using GPy. Does this by minimizing the area of the 95% confidence interval.
- GP outputs a 95% confidence interval where the most likely functions would pass through.
- We use the mean value of such functions.

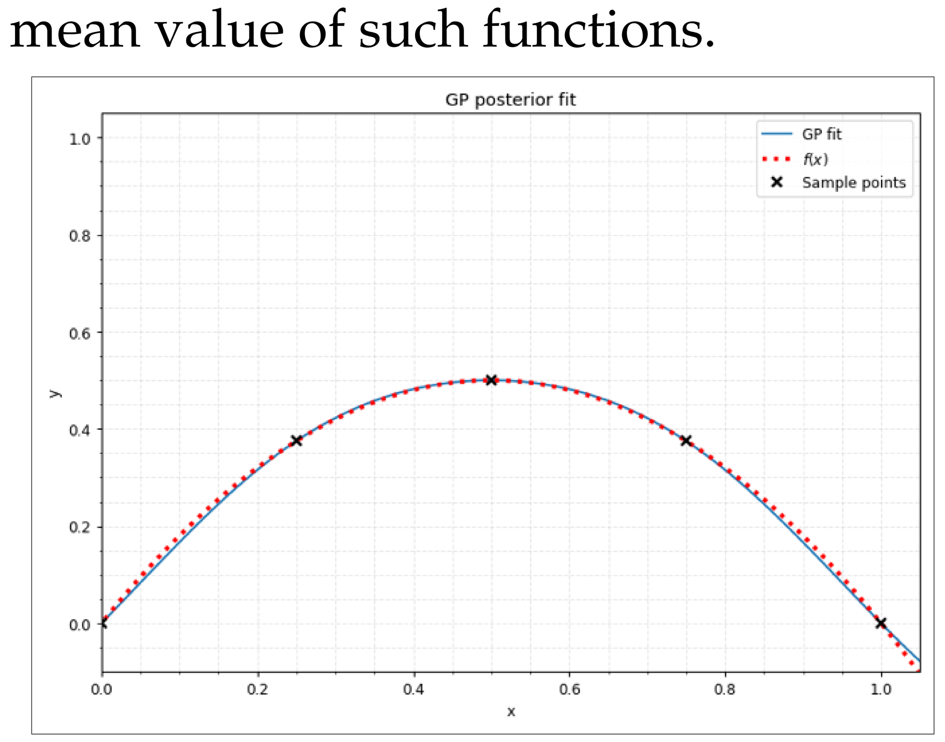

Figure 1: Using GP regression to model the logistic function

Comparing Orbits

- The orbit of the logistic map f is the sequence $\{x_0, f(x_0), f^2(x_0), \cdots\} = \{x_0, x_1, x_2, \cdots\}$. This can be compactly written as a function $x_{n+1} = f(x_n)$.
- x^* is fixed point if $f(x^*) = x^*$.
- A bifurcation happens when fixed points are created, destroyed, or their stability changes. This means a fixed point becomes stable or unstable, or it becomes a cycle or changes cycle type when the orbit changes periodicity [2].
- Orbit diagrams let us visualize the attractors the orbits of a map go to as we vary r, after some iterations.

Figure 2 shows an overlay of the orbit diagrams for the logistic map and for the GP fit. For this figure, the GP fit is constructed from $N = 10$ evenly spaced sample points of the logistic map in the interval $[0, 1]$ with no observational or process noise.

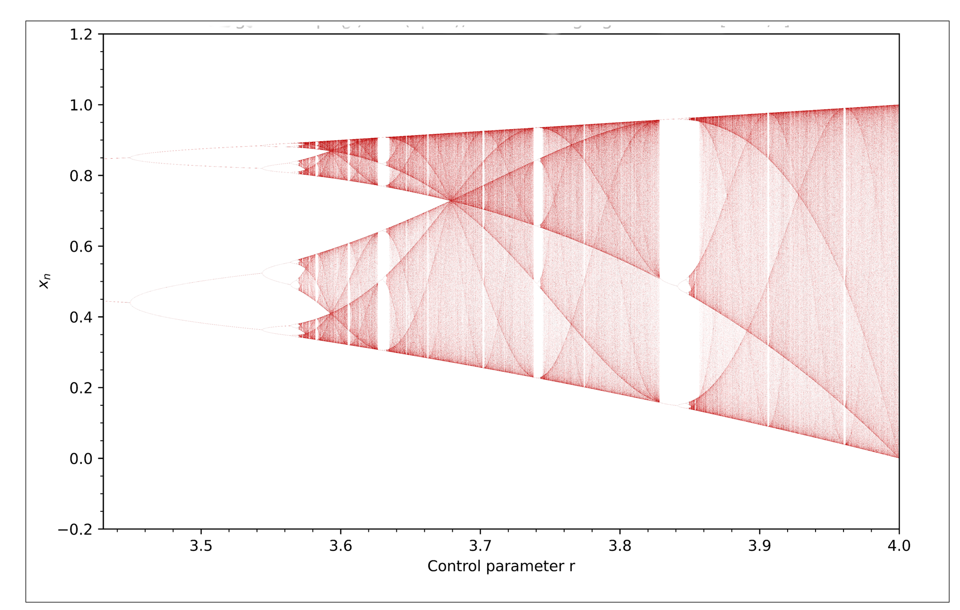

Figure 2: Overlay of the logistic map's orbit diagram, in red, with the GP fit's orbit diagram, in grey.

Figure 3 shows the two images overlaid on one another zoomed-in. We see that the orbit diagram of the GP fit hits the bifurcation points earlier than the logistic map. We see this behavior for GP constructions with no observational or process noise for other N amount of uniformly spaced points.

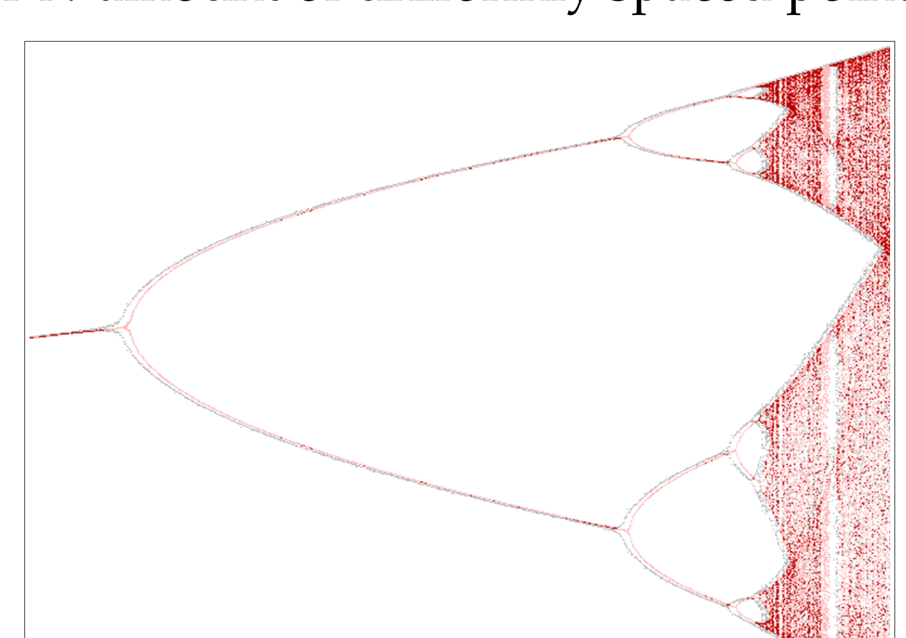

Figure 3: Same plot as above, but zoomed-in. We can see that the GP fit hits the bifurcation points earlier than the logistic map it models.

Results

- To compare orbit diagrams we use the Wasserstein distance metric, a way of measuring the distance between two distributions of points. It represents the required work to move one set of points to another [1].
- We plot the Wasserstein distance between the two orbit diagrams as we vary $r \in [0, 4]$.

See Figure 4 for an example of the Wasserstein distance between the orbit diagrams of the logistic map and GP fit.

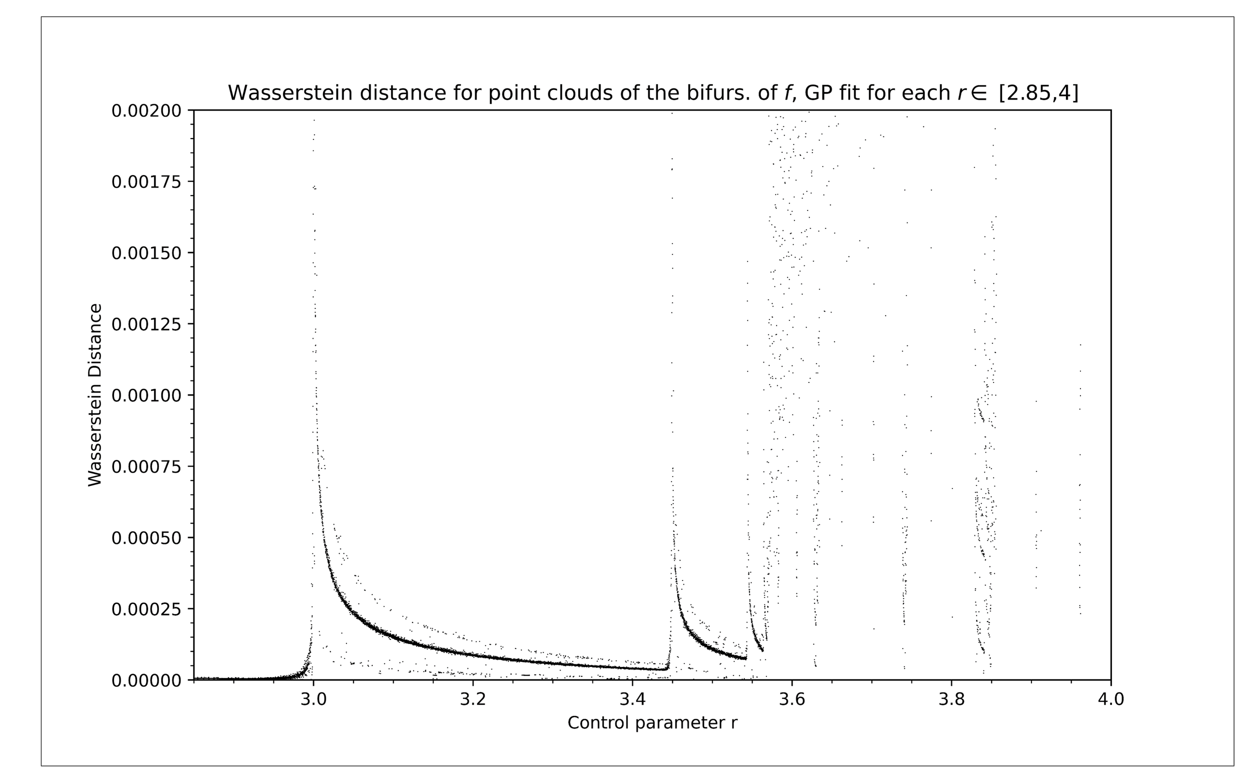

Figure 4: Measuring the Wasserstein distance between the GP fit point cloud and the logistic map point cloud.

Conclusion

- We observe these "ghost" points above and below the main Wasserstein distance curve.
- Note that the exact bifurcation values of the two different period doubling cascades may shift, which is confirmed in the Wasserstein distance plots.
- Implications for studying dynamics from time series data since a reconstruction of a system's period doubling cascade may lead to unpredictable model behavior in the neighborhood of the true bifurcation parameter value.

Refrences

1. Fisher, Robert. "The Earth Mover's Distance." *EMD*
2. Strogatz, Steven H. *Nonlinear Dynamics and Chaos: With Applications to Physics, Biology, Chemistry, and Engineering*. CRC Press, 2018.