# PROJECT 03

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## Contents



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# 1. Dehn Twists about Bounding Pairs

**1.**

- $T_{\gamma_1}$ : $\alpha_1 \mapsto \alpha_1$ $\beta_1 \mapsto \beta_1\alpha_1$
  $\alpha_2 \mapsto \alpha_2$ $\beta_2 \mapsto \beta_2$
  $\alpha_3 \mapsto \alpha_3$ $\beta_3 \mapsto \beta_3$

- $T_{\gamma_2}$ : $\alpha_1 \mapsto \alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\alpha_1[\alpha_3,\beta_3]\alpha_1$ $\beta_1 \mapsto \beta_1[\alpha_3,\beta_3]\alpha_1$
  $\alpha_2 \mapsto \alpha_2$ $\beta_2 \mapsto \beta_2$
  $\alpha_3 \mapsto \alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\alpha_3[\alpha_3,\beta_3]\alpha_1$ $\beta_3 \mapsto \alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\beta_3[\alpha_3,\beta_3]\alpha_1$

- $T_{\gamma_2}^{-1}$ : $\alpha_1 \mapsto [\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]^{-1}$ $\beta_1 \mapsto \beta_1\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}$
  $\alpha_2 \mapsto \alpha_2$ $\beta_2 \mapsto \beta_2$
  $\alpha_3 \mapsto [\alpha_3,\beta_3]\alpha_1\alpha_3\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}$ $\beta_3 \mapsto [\alpha_3,\beta_3]\alpha_1\beta_3\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}$

- $T_{\gamma_1}\circ T_{\gamma_2}^{-1}$ : $\alpha_1 \mapsto [\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]^{-1}$ $\beta_1 \mapsto \beta_1[\alpha_3,\beta_3]^{-1}$
  $\alpha_2 \mapsto \alpha_2$ $\beta_2 \mapsto \beta_2$
  $\alpha_3 \mapsto [\alpha_3,\beta_3]\alpha_1\alpha_3\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}$ $\beta_3 \mapsto [\alpha_3,\beta_3]\alpha_1\beta_3\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}$

## 1.1. Fixed Point Equations for $\left(T_{\gamma_1}\circ T_{\gamma_2}^{-1}\right)^*$.

1.1. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .

- On $R(\Sigma,G)$ : $A_1 \mapsto [A_3,B_3]A_1[A_3,B_3]^{-1}$ $B_1 \mapsto B_1[A_3,B_3]^{-1}$
  $A_2 \mapsto A_2$ $B_2 \mapsto B_2$
  $A_3 \mapsto [A_3,B_3]A_1A_3A_1^{-1}[A_3,B_3]^{-1}$ $B_3 \mapsto [A_3,B_3]A_1B_3A_1^{-1}[A_3,B_3]^{-1}$

- On $R(\Sigma,G)/G$ : $T^{-1}A_1T \mapsto [A_3,B_3]A_1[A_3,B_3]^{-1}$ $T^{-1}B_1T \mapsto B_1[A_3,B_3]^{-1}$
  $T^{-1}A_2T \mapsto A_2$ $T^{-1}B_2T \mapsto B_2$
  $T^{-1}A_3T \mapsto [A_3,B_3]A_1A_3A_1^{-1}[A_3,B_3]^{-1}$ $T^{-1}B_3T \mapsto [A_3,B_3]A_1B_3A_1^{-1}[A_3,B_3]^{-1}$

## 1.2. Checking Relation.

1.2. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .

- $T_{\gamma_1}$ :

$$\left[\alpha_1,\ \beta_1\alpha_1\right]\left[\alpha_2,\beta_2\right]\left[\alpha_3,\beta_3\right]$$
$$\alpha_1\beta_1\alpha_1\alpha_1^{-1}\alpha_1^{-1}\beta_1^{-1}\left[\alpha_2,\beta_2\right]\left[\alpha_3,\beta_3\right]$$
$$\alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1}\left[\alpha_2,\beta_2\right]\left[\alpha_3,\beta_3\right]$$
$$\left[\alpha_1,\beta_1\right]\left[\alpha_2,\beta_2\right]\left[\alpha_3,\beta_3\right]$$
$$\boxed{1}$$

- $T_{\gamma_2}$ :

$$\left[\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\alpha_1[\alpha_3,\beta_3]\alpha_1,\ \beta_1[\alpha_3,\beta_3]\alpha_1\right][\alpha_2,\beta_2]\,\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3][\alpha_3,\beta_3]\alpha_1$$
$$\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\alpha_1[\alpha_3,\beta_3]\alpha_1\beta_1[\alpha_3,\beta_3]\alpha_1\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]\alpha_1\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\beta_1^{-1}[\alpha_2,\beta_2]\,\alpha_1^{-1}[\alpha_3,\beta_3]\alpha_1$$
$$\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\alpha_1[\alpha_3,\beta_3]\alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1}[\alpha_2,\beta_2][\alpha_3,\beta_3][\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]\alpha_1$$
$$\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\alpha_1[\alpha_3,\beta_3][\alpha_1,\beta_1][\alpha_2,\beta_2][\alpha_3,\beta_3][\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]\alpha_1$$
$$\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\alpha_1[\alpha_3,\beta_3][\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]\alpha_1$$
$$\boxed{1}$$

- $T_{\gamma_2}^{-1}$ :

$$\left[[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]^{-1},\ \beta_1\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\right][\alpha_2,\beta_2][\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\,\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}$$
$$[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]^{-1}\beta_1\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1\beta_1^{-1}[\alpha_2,\beta_2][\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}$$
$$[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}\alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1}[\alpha_2,\beta_2][\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}$$
$$[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}\alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1}[\alpha_2,\beta_2][\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}$$
$$[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_1,\beta_1][\alpha_2,\beta_2][\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}$$
$$[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}\alpha_1[\alpha_3,\beta_3]\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}$$
$$\boxed{1}$$

- $T_{\gamma_1}\circ T_{\gamma_2}^{-1}$ :

$$\left[[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]^{-1},\ \beta_1[\alpha_3,\beta_3]^{-1}\right][\alpha_2,\beta_2][\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\,\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}$$
$$[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]^{-1}\beta_1[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\beta_1^{-1}[\alpha_2,\beta_2][\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}$$
$$[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}\alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1}[\alpha_2,\beta_2][\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}$$
$$[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}\alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1}[\alpha_2,\beta_2][\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}$$
$$[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_1,\beta_1][\alpha_2,\beta_2][\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}$$
$$[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}\alpha_1[\alpha_3,\beta_3]\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}$$
$$\boxed{1}$$

### 1.3. **Computing** Fix($\Phi^*$).



- $A_1$ :

$$A_1 = [A_3,B_3]A_1[A_3,B_3]^{-1}$$
$$A_1 = A_1$$

- $B_1$ :

$$B_1 = B_1[A_3,B_3]^{-1}$$
$$I = [A_3,B_3]^{-1}$$
$$I = [A_3,B_3]$$

- $A_3$ :

$$A_3 = [A_3, B_3] A_1\, A_3\, A_1^{-1} [A_3, B_3]^{-1}$$
$$A_3 = A_1 A_3 A_1^{-1}$$
$$A_1 A_3 = A_3 A_1$$
$$\big[A_1, A_3\big] = I$$

- $B_3$ :

$$B_3 = [A_3, B_3] A_1\, B_3\, A_1^{-1} [A_3, B_3]^{-1}$$
$$B_3 = A_1 B_3 A_1^{-1}$$
$$A_1 B_3 = B_3 A_1$$
$$\big[A_1, B_3\big] = I$$

Therefore, with the relations

$$\big[A_3, B_3\big] = I$$
$$\big[A_1, A_3\big] = I$$
$$\big[A_1, B_3\big] = I$$

our fixed point set for the map $\Phi^* = \big(T_{\gamma_1} \circ T_{\gamma_2}^{-1}\big)^*$ is defined as follows

$$\mathrm{Fix}(\Phi^*) = \left\{ (A_1, B_1, A_2, B_2, A_3, B_3) \in \mathrm{SU}(2)^6 \;:\; \prod_{i=1}^{3} \big[A_i, B_i\big] = I,\; \big[A_1, A_3\big] = \big[A_1, B_3\big] = \big[A_3, B_3\big] = I \right\}$$

Now let us proceed with the proof that this set is connected.

**Proposition 1.** $\mathrm{Fix}(\Phi^*)$ *is connected*

*Proof.* With

$$\mathrm{Fix}(\Phi^*) = \left\{ (A_1, B_1, A_2, B_2, A_3, B_3) \in \mathrm{SU}(2)^6 \;:\; \prod_{i=1}^{3} \big[A_i, B_i\big] = I,\; \big[A_1, A_3\big] = \big[A_1, B_3\big] = \big[A_3, B_3\big] = I \right\}$$

we begin by noting that since $[A_3, B_3] = I$, then our product of the commutator relation simplifies to

$$\big[A_1, B_1\big] = \big[A_2, B_2\big]^{-1}$$

thus we may express $\mathrm{Fix}(\Phi^*)$ as follows

$$\mathrm{Fix}(\Phi^*) = \Big\{ (A_1, B_1, A_2, B_2, A_3, B_3) \in \mathrm{SU}(2)^6 \;:\; \big[A_1, B_1\big] = \big[A_2, B_2\big]^{-1},\; \big[A_1, A_3\big] = \big[A_1, B_3\big] = \big[A_3, B_3\big] = I \Big\}$$

Now, define

$$\mathcal{B} := \big\{ (A_1, B_1, A_3, B_3) \in \mathrm{SU}(2)^4 \;:\; \big[A_1, A_3\big] = \big[A_1, B_3\big] = \big[A_3, B_3\big] = I \big\}$$

and consider the map

$$\pi : \mathrm{Fix}(\Phi^*) \longrightarrow \mathcal{B}$$

given by

$$\big(A_1, B_1, A_2, B_2, A_3, B_3\big) \longmapsto \big(A_1, B_1, A_3, B_3\big)$$

We claim that our map $\pi$ is continuous and surjective. Continuity follows directly from the observation that $\pi$ is a restriction of the projection map

$$p : \mathrm{SU}(2)^6 \longrightarrow \mathrm{SU}(2)^4$$

and we know projections in the product topology are continuous. Thus it is left to show that $\pi$ is a surjection. For this let us take $\big(A_1, B_1, A_3, B_3\big) \in \mathcal{B}$. By the definition of $\mathcal{B}$, we have

$$\big[A_1, A_3\big] = I \qquad \big[A_1, B_3\big] = I \qquad \big[A_3, B_3\big] = I$$

Thus our goal is to find an $A_2, B_2 \in \mathrm{SU}(2)$ such that the 6-tuple $(A_1, B_1, A_2, B_2, A_3, B_3)$ lies in $\mathrm{Fix}(\Phi^*)$. That is

$$\big[A_1, B_1\big] = \big[A_2, B_2\big]^{-1}$$

Define

$$A_2 := A_1^{-1} \quad \text{and} \quad B_2 := B_1^{-1},$$

note that since SU(2) is a group, then $A_2, B_2 \in \mathrm{SU}(2)$. Now, observe that

$$\big[A_1, B_1\big] = A_1 B_1 A_1^{-1} B_1^{-1} = A_2^{-1} B_2^{-1} A_2 B_2 = \big[A_2, B_2\big]^{-1}$$

Hence along with the assumed commutation relations in $\mathcal{B}$, it follows that

$$\big(A_1, B_1, A_2, B_2, A_3, B_3\big) \in \mathrm{Fix}(\Phi^*)$$

and thus

$$\pi\big(A_1, B_1, A_2, B_2, A_3, B_3\big) = \big(A_1, B_1, A_3, B_3\big)$$

so $\pi$ is surjective. Now that we have established that $\pi$ is a continuous surjection, let us consider the fibers of this map. Given a point $\big(A_1, B_1, A_3, B_3\big) \in \mathcal{B}$, the fiber over this point is defined as

$$\pi^{-1}\big(A_1, B_1, A_3, B_3\big) = \big\{(A_1, B_1, A_2, B_2, A_3, B_3) \in \mathrm{Fix}(\Phi^*)\big\}$$

Note that since in $\mathrm{Fix}(\Phi^*)$ the only relation involving $\big(A_2, B_2\big)$ is the product of the commutators, which we can express as

$$\big[A_2, B_2\big] = \big[A_1, B_1\big]^{-1}$$

we may fix $\big(A_1, B_1, A_3, B_3\big) \in \mathcal{B}$ and define the map

$$\psi : \pi^{-1}\big(A_1, B_1, A_3, B_3\big) \;\longrightarrow\; \big\{(A_2, B_2) \in \mathrm{SU}(2)^2 \;:\; [A_2, B_2] = [A_1, B_1]^{-1}\big\}$$

given by

$$\big(A_1, B_1, A_2, B_2, A_3, B_3\big) \;\longmapsto\; \big(A_2, B_2\big)$$

Since we fixed our fiber, acting as the domain, by the uniqueness of inverses in SU(2) our map $\psi$ is well-defined and surjective. Moreover, by this uniqueness property and the fact that we fixed our $A_1, B_1, A_3$ and $B_3$ arguments, we must have that the map is injective, thus $\psi$ is a bijection. Now we observe that $\psi$ is a restriction of the projection map

$$\mathrm{p} : \mathrm{SU}(2)^6 \;\longrightarrow\; \mathrm{SU}(2)^2$$

which we know to be continuous as projections in the product topology are continuous, thus $\psi$ is continuous. Furthermore, $\psi^{-1}$ is a restriction of the inclusion map

$$\mathfrak{i} : \mathrm{SU}(2)^2 \;\longrightarrow\; \mathrm{SU}(2)^6$$

which we know to be continuous as inclusions in the product topology are continuous, thus $\psi^{-1}$ is continuous. Therefore, $\psi$ is a homeomorphism, that is

$$\pi^{-1}\big(A_1, B_1, A_3, B_3\big) \cong \big\{(A_2, B_2) \in \mathrm{SU}(2)^2 \;:\; [A_2, B_2] = [A_1, B_1]^{-1}\big\}$$

Importantly, observe that

$$\big\{(A_2, B_2) \in \mathrm{SU}(2)^2 \;:\; [A_2, B_2] = [A_1, B_1]^{-1}\big\}$$

is just a fiber over the commutator map of SU(2) and thus is connected (Goldman, Topological Components of Spaces of Representations). Since we have shown that an arbitrary fiber, $\pi^{-1}(A_1, B_1, A_3, B_3)$, is connected, it follows that the fibers of our map $\pi$ are connected. Therefore,

$\pi$ is a continuous, surjective map with connected fibers from our total space $\mathrm{Fix}(\Phi^*)$ into our defined base space $\mathcal{B}$. So it remains to show that $\mathcal{B}$ is connected. For this we begin by examining the relations in $\mathcal{B}$, observing that

$$\big[A_1, A_3\big] = \big[A_1, B_3\big] = \big[A_3, B_3\big] = I$$

implies that $A_1, A_3, B_3$ all commute. Recall that any two elements of SU(2) commute if and only if they lie in the same maximal torus, which in SU(2) is conjugate to the subgroup of diagonal matrices (Brocker and tom Dieck, Representations of Compact Lie Groups, Theorem IV.2.3). Let T denote a maximal torus in SU(2),

$$\mathrm{T} = \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} : \theta \in [0, 2\pi) \right\}$$

Since $A_1, A_3, B_3$ commute, there exists $g \in \mathrm{SU}(2)$ such that

$$gA_1g^{-1},\ gA_3g^{-1},\ gB_3g^{-1} \in \mathrm{T}$$

This follows from the fact that all maximal tori in SU(2) are conjugate and every element is contained in some maximal torus (Brocker and tom Dieck, Representations of Compact Lie Groups, Theorem IV.1.6). Therefore, the set of commuting 3-tuples in SU(2) is

$$\begin{aligned} \mathcal{C} &:= \big\{(a_1, a_3, b_3) \in \mathrm{SU}(2)^3 \ :\ [a_1, a_3] = [a_1, b_3] = [a_3, b_3] = I\big\} \\ &= \big\{(gt_1g^{-1}, gt_2g^{-1}, gt_3g^{-1}) \ :\ g \in \mathrm{SU}(2),\ t_1, t_2, t_3 \in \mathrm{T}\big\} \end{aligned}$$

We wish to show that $\mathcal{C}$ is connected, in order to do so we define the map

$$\Omega : \mathrm{SU}(2) \times \mathrm{T}^3 \longrightarrow \mathcal{C}$$

by

$$\Omega\big(g, (t_1, t_2, t_3)\big) = \big(gt_1g^{-1},\ gt_2g^{-1},\ gt_3g^{-1}\big).$$

Observe that the domain of the map, $\mathrm{SU}^2 \times \mathrm{T}^3$, is connected since SU(2) is connected, T is connected, and the finite product of connected spaces is connected (Munkres, Topology, Theorem 23.6). Therefore, it suffices to show that $\Omega$ is continuous and surjective, as the image of a connected space under a continuous map is connected (Munkres, Topology, Theorem 23.5). We begin by verifying the surjectivity of $\Omega$. Let $(a_1, a_3, b_3) \in \mathcal{C}$, then there exists $g \in \mathrm{SU}(2)$ such that

$$g^{-1}a_1g,\ g^{-1}a_3g,\ g^{-1}b_3g \in \mathrm{T}.$$

This follows from the fact that all maximal tori in SU(2) are conjugate and every element is contained in some maximal torus. Now, set

$$t_1 = g^{-1}a_1g, \quad t_2 = g^{-1}a_3g, \quad t_3 = g^{-1}b_3g.$$

Thus, every element of $\mathcal{C}$ is in the image of $\Omega$ and so the map is sujective. For the continuity of the map, we note that $\Omega$ is defined on group operations, multiplication and inversion, which are smooth in SU(2), and so the map is continuous. Hence, $\mathcal{C}$ is connected. Now, there are no relations involving $B_1$ in the definition of $\mathcal{B}$, therefore, for any fixed commuting 3-tuple $(A_1, A_3, B_3)$, $B_1$ can be any element in SU(2). Thus,

$$\mathcal{B} \cong \mathcal{C} \times \mathrm{SU}(2).$$

We know that SU(2) is connected and we have just show that $\mathcal{C}$ is connected, therefore, since the finite product of connected spaces is connected, it follows that $\mathcal{B}$ is connected. So we have shown that there is a continuous surjection

$$\pi : \mathrm{Fix}(\Phi^*) \ \longrightarrow\ \mathcal{B}$$

with non-empty, connected fibers. Since every fiber is non-empty and connected, and the base space $\mathcal{B}$ is connected, it follows that $\mathrm{Fix}(\Phi^*)$ is connected. $\square$

With the fixed point set corresponding to this first power of $\Phi^*$ done, we now will move on to attempt to classify higher powers of $\Phi^*$.

## 2. Computing Powers of $\Phi$, $(n = 2)$

**2.**

- $T_{\gamma_1} \circ T_{\gamma_2}^{-1}$ :

$$\begin{aligned} \alpha_1 &\mapsto [\alpha_3, \beta_3]\alpha_1[\alpha_3, \beta_3]^{-1} & \beta_1 &\mapsto \beta_1[\alpha_3, \beta_3]^{-1} \\ \alpha_2 &\mapsto \alpha_2 & \beta_2 &\mapsto \beta_2 \\ \alpha_3 &\mapsto [\alpha_3, \beta_3]\alpha_1\alpha_3\alpha_1^{-1}[\alpha_3, \beta_3]^{-1} & \beta_3 &\mapsto [\alpha_3, \beta_3]\alpha_1\beta_3\alpha_1^{-1}[\alpha_3, \beta_3]^{-1} \end{aligned}$$

- $T_{\gamma_1} \circ T_{\gamma_2}^{-1} \circ T_{\gamma_1} \circ T_{\gamma_2}^{-1}$ :

$$\begin{aligned} \alpha_1 &\mapsto [\alpha_3, \beta_3]\alpha_1[\alpha_3, \beta_3]\alpha_1[\alpha_3, \beta_3]^{-1}\alpha_1^{-1}[\alpha_3, \beta_3]^{-1} \\ \beta_1 &\mapsto \beta_1\alpha_1[\alpha_3, \beta_3]^{-1}\alpha_1^{-1}[\alpha_3, \beta_3]^{-1} \\ \alpha_2 &\mapsto \alpha_2 \\ \beta_2 &\mapsto \beta_2 \\ \alpha_3 &\mapsto [\alpha_3, \beta_3]\alpha_1[\alpha_3, \beta_3]\alpha_1\alpha_3\alpha_1^{-1}[\alpha_3, \beta_3]^{-1}\alpha_1^{-1}[\alpha_3, \beta_3]^{-1} \\ \beta_3 &\mapsto [\alpha_3, \beta_3]\alpha_1[\alpha_3, \beta_3]\alpha_1\beta_3\alpha_1^{-1}[\alpha_3, \beta_3]^{-1}\alpha_1^{-1}[\alpha_3, \beta_3]^{-1} \end{aligned}$$

### 2.1. Fixed Point Equations for $\left(T_{\gamma_1} \circ T_{\gamma_2}^{-1} \circ T_{\gamma_1} \circ T_{\gamma_2}^{-1}\right)^*$.

2.1. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .

- On $R(\Sigma, G)$ :

$$\begin{aligned} A_1 &\mapsto [A_3, B_3]A_1[A_3, B_3]A_1[A_3, B_3]^{-1}A_1^{-1}[A_3, B_3]^{-1} \\ B_1 &\mapsto B_1A_1[A_3, B_3]^{-1}A_1^{-1}[A_3, B_3]^{-1} \\ A_2 &\mapsto A_2 \\ B_2 &\mapsto B_2 \\ A_3 &\mapsto [A_3, B_3]A_1[A_3, B_3]A_1A_3A_1^{-1}[A_3, B_3]^{-1}A_1^{-1}[A_3, B_3]^{-1} \\ B_3 &\mapsto [A_3, B_3]A_1[A_3, B_3]A_1B_3A_1^{-1}[A_3, B_3]^{-1}A_1^{-1}[A_3, B_3]^{-1} \end{aligned}$$

- On $R(\Sigma, G)/G$ :

$$\begin{aligned} T^{-1}A_1T &\mapsto [A_3, B_3]A_1[A_3, B_3]A_1[A_3, B_3]^{-1}A_1^{-1}[A_3, B_3]^{-1} \\ T^{-1}B_1T &\mapsto B_1A_1[A_3, B_3]^{-1}A_1^{-1}[A_3, B_3]^{-1} \\ T^{-1}A_2T &\mapsto A_2 \\ T^{-1}B_2T &\mapsto B_2 \\ T^{-1}A_3T &\mapsto [A_3, B_3]A_1[A_3, B_3]A_1A_3A_1^{-1}[A_3, B_3]^{-1}A_1^{-1}[A_3, B_3]^{-1} \\ T^{-1}B_3T &\mapsto [A_3, B_3]A_1[A_3, B_3]A_1B_3A_1^{-1}[A_3, B_3]^{-1}A_1^{-1}[A_3, B_3]^{-1} \end{aligned}$$

### 2.2. Computing $\Phi^2 = T_{\gamma_1} \circ T_{\gamma_2}^{-1} \circ T_{\gamma_1} \circ T_{\gamma_2}^{-1}$.

2.2. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .

- $\Phi^2(\alpha_1)$ :

$$\begin{aligned}
\Phi(\alpha_1) &= [\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]^{-1}\\
\Phi\big(\Phi(\alpha_1)\big) &= \Phi([\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]^{-1})\\
&= \Phi([\alpha_3,\beta_3])\Phi(\alpha_1)\Phi([\alpha_3,\beta_3]^{-1})\\
&= \Phi([\alpha_3,\beta_3])\Phi(\alpha_1)\big(\Phi([\alpha_3,\beta_3])\big)^{-1}\\
&= \Phi(\alpha_3\beta_3\alpha_3^{-1}\beta_3^{-1})\Phi(\alpha_1)\big(\Phi(\alpha_3\beta_3\alpha_3^{-1}\beta_3^{-1})\big)^{-1}\\
&= \Phi(\alpha_3)\Phi(\beta_3)\Phi(\alpha_3^{-1})\Phi(\beta_3^{-1})\Phi(\alpha_1)\big(\Phi(\alpha_3)\Phi(\beta_3)\Phi(\alpha_3^{-1})\Phi(\beta_3^{-1})\big)^{-1}\\
&= \Phi(\alpha_3)\Phi(\beta_3)\big(\Phi(\alpha_3)\big)^{-1}\big(\Phi(\beta_3)\big)^{-1}\Phi(\alpha_1)\big(\Phi(\alpha_3)\Phi(\beta_3)\big(\Phi(\alpha_3)\big)^{-1}\big(\Phi(\beta_3)\big)^{-1}\big)^{-1}\\
&= [\alpha_3,\beta_3]\alpha_1\alpha_3\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1\beta_3\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1\alpha_3^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1\beta_3^{-1}\alpha_1^{-1}\\
&\quad [\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1\beta_3\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1\alpha_3\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1\beta_3^{-1}\\
&\quad \alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1\alpha_3^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\\
&= [\alpha_3,\beta_3]\alpha_1\alpha_3\beta_3\alpha_3^{-1}\beta_3^{-1}\alpha_1\beta_3\alpha_3\beta_3^{-1}\alpha_3^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\\
&= [\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1[\beta_3,\alpha_3]\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\\
&= [\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}
\end{aligned}$$

$$\boxed{\Phi^2(\alpha_1) = [\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}}$$

- $\Phi^2(\beta_1)$ :

$$\begin{aligned}
\Phi(\beta_1) &= \beta_1[\alpha_3,\beta_3]^{-1}\\
\Phi\big(\Phi(\beta_1)\big) &= \Phi(\beta_1[\alpha_3,\beta_3]^{-1})\\
&= \Phi(\beta_1)\Phi([\alpha_3,\beta_3]^{-1})\\
&= \Phi(\beta_1)\big(\Phi([\alpha_3,\beta_3])\big)^{-1}\\
&= \Phi(\beta_1)\big(\Phi(\alpha_3)\Phi(\beta_3)\Phi(\alpha_3^{-1})\Phi(\beta_3^{-1})\big)^{-1}\\
&= \Phi(\beta_1)\big(\Phi(\alpha_3)\Phi(\beta_3)\big(\Phi(\alpha_3)\big)^{-1}\big(\Phi(\beta_3)\big)^{-1}\big)^{-1}\\
&= \beta_1[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1\beta_3\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1\alpha_3\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1\beta_3^{-1}\\
&\quad \alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1\alpha_3^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\\
&= \beta_1\alpha_1\beta_3\alpha_3\beta_3^{-1}\alpha_3^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\\
&= \beta_1\alpha_1[\beta_3,\alpha_3]\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\\
&= \beta_1\alpha_1[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}
\end{aligned}$$

$$\boxed{\Phi^2(\beta_1) = \beta_1\alpha_1[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}}$$

- $\Phi^2(\alpha_2)$ :

$$\begin{aligned}
\Phi(\alpha_2) &= \alpha_2\\
\Phi\big(\Phi(\alpha_2)\big) &= \Phi(\alpha_2)\\
&= \alpha_2
\end{aligned}$$

$$\boxed{\Phi^2(\alpha_2) = \alpha_2}$$

- $\Phi^2(\beta_2)$ :

$$\begin{aligned}\Phi(\beta_2) &= \beta_2\\ \Phi\big(\Phi(\beta_2)\big) &= \Phi(\beta_2)\\ &= \beta_2\end{aligned}$$

$$\boxed{\Phi^2(\beta_2) = \beta_2}$$

- $\Phi^2(\alpha_3)$ :

$$\begin{aligned}\Phi(\alpha_3) &= [\alpha_3,\beta_3]\alpha_1\alpha_3\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\\ \Phi\big(\Phi(\alpha_3)\big) &= \Phi([\alpha_3,\beta_3]\alpha_1\alpha_3\alpha_1^{-1}[\alpha_3,\beta_3]^{-1})\\ &= \Phi([\alpha_3,\beta_3])\Phi(\alpha_1)\Phi(\alpha_3)\Phi(\alpha_1^{-1})\Phi([\alpha_3,\beta_3]^{-1})\\ &= \Phi([\alpha_3,\beta_3])\Phi(\alpha_1)\Phi(\alpha_3)\Phi(\alpha_1^{-1})\big(\Phi([\alpha_3,\beta_3])\big)^{-1}\\ &= \Phi(\alpha_3\beta_3\alpha_3^{-1}\beta_3^{-1})\Phi(\alpha_1)\Phi(\alpha_3)\Phi(\alpha_1^{-1})\big(\Phi(\alpha_3\beta_3\alpha_3^{-1}\beta_3^{-1})\big)^{-1}\\ &= \Phi(\alpha_3)\Phi(\beta_3)\Phi(\alpha_3^{-1})\Phi(\beta_3^{-1})\Phi(\alpha_1)\Phi(\alpha_3)\Phi(\alpha_1^{-1})\big(\Phi(\alpha_3)\Phi(\beta_3)\Phi(\alpha_3^{-1})\Phi(\beta_3^{-1})\big)^{-1}\\ &= \Phi(\alpha_3)\Phi(\beta_3)\big(\Phi(\alpha_3)\big)^{-1}\big(\Phi(\beta_3)\big)^{-1}\Phi(\alpha_1)\Phi(\alpha_3)\big(\Phi(\alpha_1)\big)^{-1}\big(\Phi(\alpha_3)\Phi(\beta_3)\big(\Phi(\alpha_3)\big)^{-1}\big(\Phi(\beta_3)^{-1}\big)^{-1}\\ &= [\alpha_3,\beta_3]\alpha_1\alpha_3\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1\beta_3\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1\alpha_3^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1\beta_3^{-1}\alpha_1^{-1}\\ &\quad [\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1\alpha_3\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1\beta_3\alpha_1^{-1}\\ &\quad [\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1\alpha_3\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1\beta_3^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1\alpha_3^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\\ &= [\alpha_3,\beta_3]\alpha_1\alpha_3\beta_3\alpha_3^{-1}\beta_3^{-1}\alpha_1\alpha_3\alpha_1^{-1}\beta_3\alpha_3\beta_3^{-1}\alpha_3^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\\ &= [\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1\alpha_3\alpha_1^{-1}[\beta_3,\alpha_3]\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\\ &= [\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1\alpha_3\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\end{aligned}$$

$$\boxed{\Phi^2(\alpha_3) = [\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1\alpha_3\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}}$$

- $\Phi^2(\beta_3)$ :

$$\begin{aligned}\Phi(\beta_3) &= [\alpha_3,\beta_3]\alpha_1\beta_3\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\\ \Phi\big(\Phi(\beta_3)\big) &= \Phi([\alpha_3,\beta_3]\alpha_1\beta_3\alpha_1^{-1}[\alpha_3,\beta_3]^{-1})\\ &= \Phi([\alpha_3,\beta_3])\Phi(\alpha_1)\Phi(\beta_3)\Phi(\alpha_1^{-1})\Phi([\alpha_3,\beta_3]^{-1})\\ &= \Phi([\alpha_3,\beta_3])\Phi(\alpha_1)\Phi(\beta_3)\Phi(\alpha_1^{-1})\big(\Phi([\alpha_3,\beta_3])\big)^{-1}\\ &= \Phi(\alpha_3\beta_3\alpha_3^{-1}\beta_3^{-1})\Phi(\alpha_1)\Phi(\beta_3)\Phi(\alpha_1^{-1})\big(\Phi(\alpha_3\beta_3\alpha_3^{-1}\beta_3^{-1})\big)^{-1}\\ &= \Phi(\alpha_3)\Phi(\beta_3)\Phi(\alpha_3^{-1})\Phi(\beta_3^{-1})\Phi(\alpha_1)\Phi(\beta_3)\Phi(\alpha_1^{-1})\big(\Phi(\alpha_3)\Phi(\beta_3)\Phi(\alpha_3^{-1})\Phi(\beta_3^{-1})\big)^{-1}\\ &= \Phi(\alpha_3)\Phi(\beta_3)\big(\Phi(\alpha_3)\big)^{-1}\big(\Phi(\beta_3)\big)^{-1}\Phi(\alpha_1)\Phi(\beta_3)\big(\Phi(\alpha_1)\big)^{-1}\big(\Phi(\alpha_3)\Phi(\beta_3)\big(\Phi(\alpha_3)\big)^{-1}\big(\Phi(\beta_3)^{-1}\big)^{-1}\\ &= [\alpha_3,\beta_3]\alpha_1\alpha_3\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1\beta_3\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1\alpha_3^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1\beta_3^{-1}\alpha_1^{-1}\\ &\quad [\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1\beta_3\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1\beta_3\alpha_1^{-1}\\ &\quad [\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1\alpha_3\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1\beta_3^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1\alpha_3^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\\ &= [\alpha_3,\beta_3]\alpha_1\alpha_3\beta_3\alpha_3^{-1}\beta_3^{-1}\alpha_1\beta_3\alpha_1^{-1}\beta_3\alpha_3\beta_3^{-1}\alpha_3^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\\ &= [\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1\beta_3\alpha_1^{-1}[\beta_3,\alpha_3]\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\\ &= [\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1\beta_3\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\end{aligned}$$

$$\boxed{\Phi^2(\beta_3) = [\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1\beta_3\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}}$$

### 2.3. Checking Relation.

2.3. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .

- $\Phi^2 := T_{\gamma_1} \circ T_{\gamma_2}^{-1} \circ T_{\gamma_1} \circ T_{\gamma_2}^{-1}$ :

$$\text{(i)}\quad \big[\Phi^2(\alpha_1),\Phi^2(\beta_1)\big]\big[\Phi^2(\alpha_2),\Phi^2(\beta_2)\big]\big[\Phi^2(\alpha_1),\Phi^2(\beta_1)\big]$$

$$\text{(ii)}\quad \big[[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1},\beta_1\alpha_1[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\big][\alpha_2,\beta_2]$$
$$[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1\big[\alpha_3,\beta_3\big]\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}$$

$$\text{(iii)}\quad [\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\beta_1\alpha_1[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}$$
$$[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1^{-1}\beta_1^{-1}\big[\alpha_2,\beta_2\big]$$
$$[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1\big[\alpha_3,\beta_3\big]\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}$$

$$\text{(iv)}\quad [\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}\alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1}\big[\alpha_2,\beta_2\big]$$
$$\big[\alpha_3,\beta_3\big]\alpha_1[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}$$

$$\text{(v)}\quad [\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}\alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1}\big[\alpha_2,\beta_2\big]$$
$$\big[\alpha_3,\beta_3\big]\alpha_1[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}$$

$$\text{(vi)}\quad [\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}\big[\alpha_1,\beta_1\big]\big[\alpha_2,\beta_2\big]\big[\alpha_3,\beta_3\big]$$
$$\alpha_1[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}$$

$$\text{(vii)}\quad [\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}\alpha_1[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}$$

$$\text{(viii)}\quad \boxed{1}$$

### 2.4. Computing $\mathrm{Fix}\big((\Phi\circ\Phi)^*\big)$.

2.4. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .

- $A_1$ :

$$A_1 = [A_3,B_3]A_1[A_3,B_3]\,A_1\,[A_3,B_3]^{-1}A_1^{-1}[A_3,B_3]^{-1}$$
$$A_1 = A_1A_1A_1^{-1}$$
$$A_1 = A_1$$

- $B_1$ :

$$B_1 = B_1A_1[A_3,B_3]^{-1}A_1^{-1}[A_3,B_3]^{-1}$$
$$I = A_1[A_3,B_3]^{-1}A_1^{-1}[A_3,B_3]^{-1}$$
$$[A_3,B_3]A_1[A_3,B_3] = A_1$$
$$[A_3,B_3]A_1[A_3,B_3]A_1 = A_1A_1$$
$$\big([A_3,B_3]A_1\big)^2 = \big(A_1\big)^2$$

- $A_3$ :

$$A_3 = [A_3, B_3]A_1[A_3, B_3]A_1\, A_3\, A_1^{-1}[A_3, B_3]^{-1}A_1^{-1}[A_3, B_3]^{-1}$$
$$A_3 = A_1 A_1\, A_3\, A_1^{-1}A_1^{-1}$$
$$A_3 A_1 A_1 = A_1 A_1 A_3$$
$$\big[(A_1)^2, A_3\big] = I$$

- $B_3$ :

$$B_3 = [A_3, B_3]A_1[A_3, B_3]A_1\, B_3\, A_1^{-1}[A_3, B_3]^{-1}A_1^{-1}[A_3, B_3]^{-1}$$
$$B_3 = A_1 A_1\, B_3\, A_1^{-1}A_1^{-1}$$
$$B_3 A_1 A_1 = A_1 A_1 B_3$$
$$\big[(A_1)^2, B_3\big] = I$$

Therefore, with the relations

$$\big([A_3, B_3]A_1\big)^2 = \big(A_1\big)^2$$
$$\big[(A_1)^2, A_3\big] = I$$
$$\big[(A_1)^2, B_3\big] = I$$

our fixed point set for the map $\big(\Phi \circ \Phi\big)^* = \big(T_{\gamma_1} \circ T_{\gamma_2}^{-1} \circ T_{\gamma_1} \circ T_{\gamma_2}^{-1}\big)^*$ is defined as follows:

$$\mathrm{Fix}\big((\Phi^2)^*\big) = \left\{ \big(A_i, B_i\big) \in \mathrm{SU}(2)^6 \ : \ \prod_{i=1}^{3} [A_i, B_i] = I,\ \big([A_3, B_3]A_1\big)^2 = \big(A_1\big)^2,\ \big[\big(A_1\big)^2, A_3\big] = I,\ \big[\big(A_1\big)^2, B_3\big] = I \right\}$$

Now, observe that $(A_1)^2$ is involved in three of our four relations, and that based on this $A_1$ essentially determines the other five entries. Thus define the map

$$\mu : \mathrm{Fix}\big((\Phi^2)^*\big) \ \longrightarrow\ \mathrm{SU}(2)$$

given by

$$\big(A_1, B_1, A_2, B_2, A_3, B_3\big) \ \longmapsto\ A_1$$

First, note that $\mu$ is a restriction of the projection map

$$\mathrm{p} : \mathrm{SU}(2)^6 \ \longrightarrow\ \mathrm{SU}(2)$$

which we know to be continuous as projections in the product topology are continuous, thus $\mu$ is continuous. Next we claim that $\mu$ is surjective.

**Lemma 1.** *The map*

$$\mu : \mathrm{Fix}\big((\Phi^2)^*\big) \ \longrightarrow\ \mathrm{SU}(2)$$
$$\big(A_1, B_1, A_2, B_2, A_3, B_3\big) \ \longmapsto\ A_1$$

*is surjective*

*Proof.* Let $A_1 \in \mathrm{SU}(2)$. Now we wish to find corresponding $B_1, A_2, B_2, A_3, B_3 \in \mathrm{SU}(2)$ such that the 6-tuple lies in our fixed point set, that is

$$\big(A_1, B_1, A_2, B_2, A_3, B_3\big) \in \mathrm{Fix}\big((\Phi^2)^*\big)$$

To do so let us consider the relations that these matrices must satisfy

$$\textstyle\prod_{i=1}^{3} \big[A_i, B_i\big] = I$$
$$\big([A_3, B_3]A_1\big)^2 = \big(A_1\big)^2$$
$$\big[(A_1)^2, A_3\big] = \big[(A_1)^2, B_3\big] = I$$

Observe that for the second and third relations, if we take both $A_3$ and $B_3$ to be the identity matrix then these two are satisfied. Furthermore, for the product of the commutator relation, with our choices of $A_3$ and $B_3$, this becomes

$$\big[A_1, B_1\big] = \big[A_2, B_2\big]^{-1}$$

Thus, we may choose our $B_1, A_2$ and $B_2$ such that this equality holds, for simplicity's sake again set them equal to the identity. With this 6-tuple

$$\big(A_1, I_{B_1}, I_{A_2}, I_{B_2}, I_{A_3}, I_{B_3}\big)$$

we have shown that it satisfies the relations in our fixed point set and thus for arbitrary $A_1 \in \mathrm{SU}(2)$ have found a corresponding element of our domain. Therefore, our map $\mu$ is surjective. □

Having shown that our map $\mu$ is a continuous surjection, we now wish to use it to classify our fixed point set. To do so we first define the following sets in SU(2) based on our possible values of $\big(A_1\big)^2$. Let

$$\mathcal{A}^2_{\not\pm} := \big\{A_1 \in \mathrm{SU}(2) \,:\, \big(A_1\big)^2 \neq \pm I\big\}$$

be the set of all $A_1$ whose 2-nd power is non-central in SU(2) and let

$$\mathcal{A}^2_{\pm} := \big\{A_1 \in \mathrm{SU}(2) \,:\, \big(A_1\big)^2 = \pm I\big\}$$

denote the set of all $A_1$ whose 2-nd power is central in SU(2). However, note that we can further partition this second set into the following two subsets,

$$\mathcal{A}^2_{+} := \big\{A_1 \in \mathrm{SU}(2) \,:\, \big(A_1\big)^2 = I\big\} = \big\{-I, I\big\}$$

and

$$\mathcal{A}^2_{-} := \big\{A_1 \in \mathrm{SU}(2) \,:\, \big(A_1\big)^2 = -I\big\}$$

We may note that by construction, these three sets partition SU(2) as they represent the collection of fibers of the 2-nd power map. Now, with these sets we may consider their preimages under our map $\mu$, notably,

$$\mu^{-1}\big(\mathcal{A}^2_{\not\pm}\big) = \big\{(A_1, B_1, A_2, B_2, A_3, B_3) \in \mathrm{Fix}\big((\Phi^2)^*\big) \,:\, \big(A_1\big)^2 \neq \pm I\big\}$$

and

$$\mu^{-1}\big(\mathcal{A}^2_{\pm}\big) = \big\{(A_1, B_1, A_2, B_2, A_3, B_3) \in \mathrm{Fix}\big((\Phi^2)^*\big) \,:\, \big(A_1\big)^2 = \pm I\big\}$$

which we can represent as

$$\mu^{-1}\big(\mathcal{A}^2_{\pm}\big) = \big(\mu^{-1}(I) \cup \mu^{-1}(-I)\big) \cup \mu^{-1}\big(\mathcal{A}^2_{-}\big)$$

with

$$\mu^{-1}\big(I\big) = \big\{(I, B_1, A_2, B_2, A_3, B_3) \in \mathrm{Fix}\big((\Phi^2)^*\big)\big\}$$
$$\mu^{-1}\big(-I\big) = \big\{(-I, B_1, A_2, B_2, A_3, B_3) \in \mathrm{Fix}\big((\Phi^2)^*\big)\big\}$$

and

$$\mu^{-1}\big(\mathcal{A}^2_{-}\big) = \big\{(A_1, B_1, A_2, B_2, A_3, B_3) \in \mathrm{Fix}\big((\Phi^2)^*\big) \,:\, \big(A_1\big)^2 = -I\big\}$$

Since $\mu$ is a continuous surjection and as previously noted our characterization sets, $\mathcal{A}^2_{\not\pm}$ and $\mathcal{A}^2_{\pm}$ partition SU(2), it follows that our fixed point set may be expressed as the union of these respective preimages, that is

$$\mathrm{Fix}\big((\Phi^2)^*\big) = \mu^{-1}\big(\mathcal{A}^2_{\not\pm}\big) \cup \mu^{-1}\big(\mathcal{A}^2_{\pm}\big) = \mu^{-1}\big(\mathcal{A}^2_{\not\pm}\big) \cup \big((\mu^{-1}(I) \cup \mu^{-1}(-I)) \cup \mu^{-1}\big(\mathcal{A}^2_{-}\big)\big)$$

Therefore, in order to classify the connectedness of our fixed point set, it suffices to investigate the connectedness of these preimages. First, we will consider the preimage over the collection of $A_1$ such that their 2-nd power is a non-central element of SU(2). To do so lets fix $A_1 \in \mathcal{A}^2_{\not\pm}$. Then, we will consider our original relations from the fixed point set. We begin by noting that by our last two relations, we know that $\big(A_1\big)^2$ commutes with both $A_3$ and $B_3$. Thus, by definition $A_3$ and $B_3$ are in the centralizer of $\big(A_1\big)^2$. Since $\big(A_1\big)^2$ is assumed to be a non-central element, its centralizer is a maximal torus in SU(2). It follows that both $A_3$ and $B_3$ lie in this maximal torus. Recall that every maximal torus is abelian, hence, $A_3$ and $B_3$ must commute, that is

$$\big[A_3, B_3\big] = I$$

Now we may consider the product of the commutator relation, noting that with $A_3$ and $B_3$ commuting our expression simplifies as follows

$$\begin{aligned} \textstyle\prod_{i=1}^{3} \big[A_i, B_i\big] &= I \\ \big[A_1, B_1\big]\big[A_2, B_2\big]\big[A_3, B_3\big] &= I \\ \big[A_1, B_1\big]\big[A_2, B_2\big] &= I \\ \big[A_1, B_1\big] &= \big[A_2, B_2\big]^{-1} \end{aligned}$$

Next lets examine our second relation. Observe that since the commutator of $A_3$ and $B_3$ is in the center of SU(2), then it commutes with $A_1$ and so we may distribute the exponent and realize that this relation is trivial. Therefore, for each $A_1 \in \mathcal{A}^2_{\not\pm}$ we can express its corresponding stratum of the fixed point set as

$$\{A_1\}\times\mathcal{F}^{\not\pm}_{A_1} := \{A_1\}\times\Big\{(B_1, A_2, B_2, A_3, B_3) \in \mathrm{SU}(2)^5 \;:\; \big[A_1, B_1\big] = \big[A_2, B_2\big]^{-1},\; \big[A_3, B_3\big] = \big[(A_1)^2, A_3\big] = \big[(A_1)^2, B_3\big] = I\Big\}$$

Hence, the preimage $\mu^{-1}\big(\mathcal{A}^2_{\not\pm}\big)$ can be represented by the union of over all such $A_1 \in \mathcal{A}^2_{\not\pm}$ of these corresponding fixed point stratum sets, that is

$$\mu^{-1}\big(\mathcal{A}^2_{\not\pm}\big) = \bigcup_{A_1\in\mathcal{A}^2_{\not\pm}} \{A_1\}\times\mathcal{F}^{\not\pm}_{A_1}$$

Next, we will consider the preimage over the collection $A_1$ such that their 2-nd power is a central element of SU(2). To do so lets fix $A_1 \in \mathcal{A}^2_{\pm}$. Then, we will consider our original relations from the fixed point set. We begin by considering the product of the commutators relation, noting that we cannot simplify this with the extra condition on $A_1$, and so we move on. Next lets examine our second relation. Observe that by our intial assumption on $A_1$ this relation simplifies to

$$\big([A_3, B_3]A_1\big)^2 = \pm I$$

Finally, examining our last two relations, since $(A_1)^2$ commutes with every element of SU(2), then these relations are trivial. Thus for each $A_1 \in \mathcal{A}^2_{\pm}$ we can express its corresponding stratum of the fixed point set as

$$\{A_1\}\times\mathcal{F}^{\pm}_{A_1} := \{A_1\}\times\left\{(B_1, A_2, B_2, A_3, B_3) \in \mathrm{SU}(2)^5 \;:\; \prod_{i=1}^{3}\big[A_i, B_i\big] = I,\; \big([A_3, B_3]A_1\big)^2 = \pm I\right\}$$

However, as we previously observed, we can split $\mathcal{A}^2_{\pm}$ into two subsets, $\mathcal{A}^2_{+}$ and $\mathcal{A}^2_{-}$. Therefore, for each $A_1' \in \mathcal{A}^2_{+}$ we can express its corresponding stratum of the fixed point set as

$$\{A_1'\}\times\mathcal{F}^{+}_{A_1'} := \{A_1'\}\times\left\{(B_1, A_2, B_2, A_3, B_3) \in \mathrm{SU}(2)^5 \;:\; \prod_{i=1}^{3}\big[A_i, B_i\big] = I,\; \big([A_3, B_3]A_1'\big)^2 = I\right\}$$

Likewise, for each $A_1'' \in \mathcal{A}^2_{-}$ we can express its corresponding stratum of the fixed point set as

$$\{A_1''\}\times\mathcal{F}^{-}_{A_1''} := \{A_1''\}\times\left\{(B_1, A_2, B_2, A_3, B_3) \in \mathrm{SU}(2)^5 \;:\; \prod_{i=1}^{3}\big[A_i, B_i\big] = I,\; \big([A_3, B_3]A_1''\big)^2 = -I\right\}$$

Now, note that these two fixed point stratum sets are disjoint as their corresponding characterization sets $\mathcal{A}^2_{+}$ and $\mathcal{A}^2_{-}$ are disjoint and so the $A_1$ arguments of each respective set cannot agree. Hence, the preimage $\mu^{-1}\big(\mathcal{A}^2_{\pm}\big)$ can be represented as the disjoint union of the two respective unions over all such $A_1' \in \mathcal{A}^2_{+}$ and $A_1'' \in \mathcal{A}^2_{-}$ of these corresponding fixed point stratum sets, that is

$$\mu^{-1}\big(\mathcal{A}^2_{\pm}\big) = \bigcup_{A_1'\in\mathcal{A}^2_{+}} \{A_1'\}\times\mathcal{F}^{+}_{A_1'} \;\sqcup\; \bigcup_{A_1''\in\mathcal{A}^2_{-}} \{A_1''\}\times\mathcal{F}^{-}_{A_1''}$$

Now that we have described each respective preimage of $\mu$ which partition our fixed point set. We may observe that our fixed point set can be represented as the disjoint union of three respective unions of our fixed point stratum sets, $\{A_1\}\times\mathcal{F}^{\not\equiv}_{A_1}$, $\{A_1'\}\times\mathcal{F}^{+}_{A_1'}$, and $\{A_1''\}\times\mathcal{F}^{-}_{A_1''}$, over our $A_1$ characterization sets, $\mathcal{A}^2_{\not\equiv}$, $\mathcal{A}^2_{+}$, and $\mathcal{A}^2_{-}$. That is, from our original representation

$$\mathrm{Fix}\big((\Phi^2)^*\big) = \mu^{-1}\big(\mathcal{A}^2_{\not\equiv}\big) \cup \big((\mu^{-1}(I)\cup\mu^{-1}(-I))\cup\mu^{-1}\big(\mathcal{A}^2_{-}\big)\big)$$

we have that

$$\mathrm{Fix}\big((\Phi^2)^*\big) = \bigcup_{A_1\in\mathcal{A}^2_{\not\equiv}}\{A_1\}\times\mathcal{F}^{\not\equiv}_{A_1} \;\cup\; \{I\}\times\mathcal{F}^{+}_{I} \;\cup\; \{-I\}\times\mathcal{F}^{+}_{-I} \;\cup \bigcup_{A_1''\in\mathcal{A}^2_{-}}\{A_1''\}\times\mathcal{F}^{-}_{A_1''}$$

Due note that this is technically a disjoint union of these respective unions as by construction the $A_1$ argument of each respective collection of fixed point stratum sets cannot agree. Therefore, as we attempt to classify the connectedness of our fixed point set, it suffices to determine the connectedness of each collection of fixed point stratum sets respectively. We will begin by investigating the connectedness of the preimage of our characterization set $\mathcal{A}^2_{\not\equiv}$, that is

$$\mu^{-1}\big(\mathcal{A}^2_{\not\equiv}\big) = \bigcup_{A_1\in\mathcal{A}^2_{\not\equiv}}\{A_1\}\times\mathcal{F}^{\not\equiv}_{A_1}$$

To do so we first consider each individual fixed point stratum set $\{A_1\}\times\mathcal{F}^{\not\equiv}_{A_1}$.

**Lemma 2.** *$\{A_1\}\times\mathcal{F}^{\not\equiv}_{A_1}$ is connected for every $A_1\in\mathcal{A}^2_{\not\equiv}$*

*Proof.* Fix $A_1\in\mathcal{A}^2_{\not\equiv}$. With

$$\{A_1\}\times\mathcal{F}^{\not\equiv}_{A_1} := \{A_1\}\times\Big\{(B_1,A_2,B_2,A_3,B_3)\in\mathrm{SU}(2)^5 \,:\, \big[A_1,B_1\big]=\big[A_2,B_2\big]^{-1},\ \big[A_3,B_3\big]=\big[(A_1)^2,A_3\big]=\big[(A_1)^2,B_3\big]=I\Big\}$$

we begin by noting that the singleton set $\{A_1\}$ is connected in SU(2) and the product

$$\{A_1\}\times\mathcal{F}^{\not\equiv}_{A_1}$$

is homeomorphic to $\mathcal{F}^{\not\equiv}_{A_1}$, as it is just a copy of this set at $A_1$. Thus to determine the connectedness of the fixed point stratum set, it suffices to show the connectedness of $\mathcal{F}^{\not\equiv}_{A_1}$. With this, we first define

$$\mathcal{B}^{\not\equiv}_{A_1} := \big\{(B_1,A_3,B_3)\in\mathrm{SU}(2)^3 \,:\, \big[A_3,B_3\big]=\big[(A_1)^2,A_3\big]=\big[(A_1)^2,B_3\big]=I\big\}$$

Now consider the map

$$\pi:\mathcal{F}^{\not\equiv}_{A_1}\longrightarrow\mathcal{B}^{\not\equiv}_{A_1}$$

given by

$$\big(B_1,A_2,B_2,A_3,B_3\big)\longmapsto\big(B_1,A_3,B_3\big)$$

Note that $\pi$ is a restriction of the projection map

$$\mathrm{p}:\mathrm{SU}(2)^5\longrightarrow\mathrm{SU}(2)^3$$

which we know to be continuous as projections in the product topology are continuous, thus $\pi$ is continuous. Additionally, we claim that $\pi$ is surjective. To see this, take $\big(B_1,A_3,B_3\big)\in\mathcal{B}^{\not\equiv}_{A_1}$. By the definition of $\mathcal{B}^{\not\equiv}_{A_1}$, we have

$$\big[A_3,B_3\big]=I \qquad \big[(A_1)^2,A_3\big]=I \qquad \big[(A_1)^2,B_3\big]=I$$

Thus our goal is to find an $A_2,B_2\in\mathrm{SU}(2)$ such that the 5-tuple $\big(B_1,A_2,B_2,A_3,B_3\big)$ lies in $\mathcal{F}^{\not\equiv}_{A_1}$. That is

$$\big[A_1,B_1\big]=\big[A_2,B_2\big]^{-1}$$

Define

$$A_2:=A_1^{-1} \quad\text{and}\quad B_2:=B_1^{-1},$$

note that since SU(2) is a group, then $A_2, B_2 \in \mathrm{SU}(2)$. Now, observe that

$$\big[A_1, B_1\big] = A_1 B_1 A_1^{-1} B_1^{-1} = A_2^{-1} B_2^{-1} A_2 B_2 = \big[A_2, B_2\big]^{-1}$$

Hence along with the assumed commutation relations in $\mathcal{B}_{A_1}^{\neq}$, it follows that

$$\big(B_1, A_2, B_2, A_3, B_3\big) \in \mathcal{F}_{A_1}^{\neq}$$

and thus

$$\pi\big(B_1, A_2, B_2, A_3, B_3\big) = \big(B_1, A_3, B_3\big)$$

so $\pi$ is surjective. Now that we have established that $\pi$ is a continuous surjection, let us consider the fibers of this map. Given a point $\big(B_1, A_3, B_3\big) \in \mathcal{B}_{A_1}^{\neq}$, the fiber over this point is defined as

$$\pi^{-1}\big(B_1, A_3, B_3\big) = \big\{(B_1, A_2, B_2, A_3, B_3) \in \mathcal{F}_{A_1}^{\neq}\big\}$$

Note that since in $\mathcal{F}_{A_1}^{\neq}$ the only relation involving $\big(A_2, B_2\big)$ is

$$\big[A_2, B_2\big] = \big[A_1, B_1\big]^{-1}$$

we may fix $\big(B_1, A_3, B_3\big) \in \mathcal{B}_{A_1}^{\neq}$ and define the map

$$\psi : \pi^{-1}\big(B_1, A_3, B_3\big) \longrightarrow \big\{(A_2, B_2) \in \mathrm{SU}(2)^2 \,:\, [A_2, B_2] = [A_1, B_1]^{-1}\big\}$$

given by

$$\big(B_1, A_2, B_2, A_3, B_3\big) \longmapsto \big(A_2, B_2\big)$$

Since we fixed our fiber, acting as the domain, by the uniqueness of inverses in SU(2) our map $\psi$ is well-defined and surjective. Moreover, by this uniqueness property and the fact that we fixed our $B_1, A_3$ and $B_3$ arguments, we must have that the map is injective, thus $\psi$ is a bijection. Now we observe that $\psi$ is a restriction of the projection map

$$\mathrm{p} : \mathrm{SU}(2)^5 \longrightarrow \mathrm{SU}(2)^2$$

which we know to be continuous as projections in the product topology are continuous, thus $\psi$ is continuous. Furthermore, $\psi^{-1}$ is a restriction of the inclusion map

$$\mathfrak{i} : \mathrm{SU}(2)^2 \longrightarrow \mathrm{SU}(2)^5$$

which we know to be continuous as inclusions in the product topology are continuous, thus $\psi^{-1}$ is continuous. Therefore, $\psi$ is a homeomorphism, that is

$$\pi^{-1}\big(B_1, A_3, B_3\big) \cong \big\{(A_2, B_2) \in \mathrm{SU}(2)^2 \,:\, [A_2, B_2] = [A_1, B_1]^{-1}\big\}.$$

Importantly, observe that

$$\big\{(A_2, B_2) \in \mathrm{SU}(2)^2 \,:\, [A_2, B_2] = [A_1, B_1]^{-1}\big\}$$

is just a fiber over the commutator map of SU(2) and thus is connected. Since we have shown that an arbitrary fiber, $\pi^{-1}(B_1, A_3, B_3)$, is connected, it follows that the fibers of our map $\pi$ are connected. Therefore, $\pi$ is a continuous, surjective map with connected fibers from our total space $\mathcal{F}_{A_1}^{\neq}$ into our defined base space $\mathcal{B}_{A_1}^{\neq}$. Thus, we will now investigate the connectedness of this base space $\mathcal{B}_{A_1}^{\pm}$. To do so let us first examine the commutation relations in $\mathcal{B}_{A_1}^{\neq}$, noting that

$$\big[A_3, B_3\big] = \big[(A_1)^2, A_3\big] = \big[(A_1)^2, B_3\big] = I$$

implies that $A_3, B_3$ commute. Recall that any two elements of SU(2) commute if and only if they lie in the same maximal torus, which in SU(2) is conjugate to the subgroup of diagonal matrices. Let T denote a maximal torus in SU(2),

$$\mathrm{T} = \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} : \theta \in [0, 2\pi) \right\}$$

Since $A_3, B_3$ commute, there exists $g \in \mathrm{SU}(2)$ such that

$$g\,A_3 g^{-1},\ g\,B_3 g^{-1} \in \mathrm{T}$$

This follows from the fact that all maximal tori in SU(2) are conjugate and every element is contained in some maximal torus. Therefore, the set of commuting 2-tuples in SU(2) is

$$\begin{aligned} \mathcal{C}^{\neq}_{A_1} &:= \left\{(a_3, b_3) \in \mathrm{SU}(2)^2 \,:\, [a_3, b_3] = I\right\} \\ &= \left\{(g\,t_1 g^{-1}, g\,t_2 g^{-1}) \,:\, g \in \mathrm{SU}(2),\ t_1, t_2 \in \mathrm{T}\right\} \end{aligned}$$

We wish to show that $\mathcal{C}^{\neq}_{A_1}$ is connected, in order to do so we define the map

$$\Omega : \mathrm{SU}(2) \times \mathrm{T}^2 \ \longrightarrow\ \mathcal{C}^{\neq}_{A_1}$$

by

$$\Omega\big(g, (t_1, t_2)\big) = \big(g\,t_1 g^{-1},\ g\,t_2 g^{-1}\big).$$

Observe that the domain of the map, $\mathrm{SU}(2) \times \mathrm{T}^2$, is connected since SU(2) is connected, T is connected, and the finite product of connected spaces is connected. Therefore, it suffices to show that $\Omega$ is continuous and surjective, as the image of a connected space under a continuous map is connected. We begin by verifying the surjectivity of $\Omega$. Let $(a_3, b_3) \in \mathcal{C}^{\neq}_{A_1}$, then there exists $g \in \mathrm{SU}(2)$ such that

$$g^{-1} a_3 g,\ g^{-1} b_3 g \in \mathrm{T}.$$

This follows from the fact that all maximal tori in SU(2) are conjugate and every element is contained in some maximal torus. Now, if we set

$$t_1 = g^{-1} a_3 g \qquad \text{and} \qquad t_2 = g^{-1} b_3 g.$$

then every element of $\mathcal{C}^{\neq}_{A_1}$ is in the image of $\mathrm{SU}(2) \times \mathrm{T}^2$ under $\Omega$ and so the map is surjective. For the continuity of the map, we note that $\Omega$ is defined on group operations, multiplication and inversion, which are smooth in SU(2), and so the map is continuous. Hence, $\mathcal{C}^{\neq}_{A_1}$ is connected. Now, there are no relations involving $B_1$ in the definition of $\mathcal{B}^{\neq}_{A_1}$, therefore, for any fixed commuting 2-tuple $\big(A_3, B_3\big)$, $B_1$ can be any element in SU(2). Thus,

$$\mathcal{B}^{\neq}_{A_1} \cong \mathcal{C}^{\neq}_{A_1} \times \mathrm{SU}(2).$$

We know that SU(2) is connected and we have just shown that $\mathcal{C}^{\neq}_{A_1}$ is connected, therefore, since the finite product of connected spaces is connected, it follows that $\mathcal{B}^{\neq}_{A_1}$ is connected. To recap, we have shown that there is a continuous surjection

$$\pi : \mathcal{F}^{\neq}_{A_1} \ \longrightarrow\ \mathcal{B}^{\neq}_{A_1}$$

with non-empty, connected fibers. Therefore, since the base space $\mathcal{B}^{\neq}_{A_1}$ is connected, it follows that $\mathcal{F}^{\neq}_{A_1}$ is connected. Hence the entire fixed point stratum set

$$\{A_1\} \times \mathcal{F}^{\neq}_{A_1}$$

is connected. □

We have now shown that for our preimage over the characterization set $\mathcal{A}^2_{\neq}$ each fixed point stratum set is connected. So to determine the connectedness of $\mu^{-1}\big(\mathcal{A}^2_{\neq}\big)$ we need to consider the union of all such fixed point stratum sets over our characterization set.

**Lemma 3.** *$\mu^{-1}\big(\mathcal{A}^2_{\neq}\big) = \bigcup_{A_1 \in \mathcal{A}^2_{\neq}} \{A_1\} \times \mathcal{F}^{\neq}_{A_1}$ has 2 connected components*

*Proof.* We begin by noting that by the previous lemma, we know that for each $A_1 \in \mathcal{A}^2_{\not\equiv}$ our corresponding fixed point stratum set

$$\{A_1\}\times\mathcal{F}^{\not\equiv}_{A_1} := \{A_1\}\times\Big\{(B_1, A_2, B_2, A_3, B_3) \in \mathrm{SU}(2)^5 \;:\; [A_1, B_1] = [A_2, B_2]^{-1},\; [A_3, B_3] = [(A_1)^2, A_3] = [(A_1)^2, B_3] = I\Big\}$$

is connected. Thus the remaining determination of the connectedness of our preimage

$$\mu^{-1}(\mathcal{A}^2_{\not\equiv}) = \bigcup_{A_1\in\mathcal{A}^2_{\not\equiv}} \{A_1\} \times \mathcal{F}^{\not\equiv}_{A_1}$$

depends on that of the parameter space, our characterization set $\mathcal{A}^2_{\not\equiv}$. Now with

$$\mathcal{A}^2_{\not\equiv} = \{A_1 \in \mathrm{SU}(2) \;:\; (A_1)^2 \neq \pm I\}$$

let us recall that this is merely the union of fibers of the 2-nd power map of SU(2). Thus we will consider this map,

$$\mathrm{p}_2 : \mathrm{SU}(2) \longrightarrow \mathrm{SU}(2)$$

given by

$$Z \longmapsto Z^2$$

Observe that since $\mathrm{p}_2$ is surjective

$$\mathcal{A}^2_{\not\equiv} = \mathrm{SU}(2) \setminus \big(\mathrm{p}_2^{-1}(I) \cup \mathrm{p}_2^{-1}(-I)\big)$$

Thus let us examine

$$\mathrm{SU}(2) \setminus \big(\mathrm{p}_2^{-1}(I) \cup \mathrm{p}_2^{-1}(-I)\big)$$

Note that for any $W \in \mathrm{SU}(2) \setminus \big(\mathrm{p}_2^{-1}(I) \cup \mathrm{p}_2^{-1}(-I)\big)$, $W$ is diagonalizable and can be written up to conjugation as

$$W \sim \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}, \quad \theta \in [0, \pi]$$

Now this represents the conjugacy class determined by the eigenvalues $(e^{i\theta}, e^{-i\theta})$, which by our condition on $W$ that

$$(W)^2 \neq I$$

must satisfy

$$(e^{i\theta})^2 \neq \pm 1 \implies e^{i2\theta} \neq \pm 1 \iff 2\theta \not\equiv 0 \pmod{\pi}$$

Therefore, we have

$$\theta \neq \frac{\pi k}{2}, \quad k \in \mathbb{Z}$$

Due to the equivalence under conjugation of $\theta \sim -\theta$ and $\theta \sim \theta + 2\pi$ in SU(2), we only consider $\theta \in [0, \pi]$. Removing all such angles from our interval $[0, \pi]$ for $0 \leq k \leq 2$ we are left with 2 open intervals

$$\left(0, \frac{\pi}{2}\right) \quad \text{and} \quad \left(\frac{\pi}{2}, \frac{2\pi}{2}\right)$$

Now, we claim that each of these open intervals corresponds to a connected component of

$$\mathrm{SU}(2) \setminus \big(\mathrm{p}_2^{-1}(I) \cup \mathrm{p}_2^{-1}(-I)\big)$$

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[INSERT] DR. DUNCAN: Implicit/Inverse Function Theorem Argument

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□

With this we have shown that $\mu^{-1}\big(\mathcal{A}^2_{\neq}\big)$ has two connected components. Thus it is left to classify the connectedness of our two remaining preimages which partition our fixed point set. To do so, first we will consider our fixed point stratum sets which arise in the case where the 2-nd power of $A_1$ is in the center of SU(2), specifically when it is equal to the identity.

**Lemma 4.** *For every $A_1 \in \mathcal{A}^2_+$, $\{A_1\} \times \mathcal{F}^+_{A_1}$ has two connected components*

*Proof.* Let us begin by noting that in SU(2) the only matrices which square to the identity matrix are $\pm I$. Thus our characterization set

$$\mathcal{A}^2_+ = \big\{A_1 \in \mathrm{SU}(2) \ : \ \big(A_1\big)^2 = I\big\} = \big\{-I, I\big\}$$

Thus we will consider the two cases for our possible $A_1$ separately. First, take $A_1 = I$. Then with

$$\{I\} \times \mathcal{F}^+_I := \big\{I\big\} \times \Big\{(B_1, A_2, B_2, A_3, B_3) \in \mathrm{SU}(2)^5 \ : \ \big[A_2, B_2\big] = \big[A_3, B_3\big]^{-1}, \ \big([A_3, B_3]\big)^2 = I\Big\}$$

we begin by noting that the singleton set $\{I\}$ is connected in SU(2) and the product

$$\{I\} \times \mathcal{F}^+_I$$

is homeomorphic to $\mathcal{F}^+_I$ as it is just a copy of this set at $I$. Thus to determine the connectedness of the fixed point stratum set, it suffices to show the connectedness of $\mathcal{F}^+_I$. Now notice that in taking $A_1 = I$ along with the simplification of the product of the commutator relation, our second relation becomes

$$\big[A_3, B_2\big] = \pm I$$

as again the only matrices in SU(2) which square to the identity matrix are $\pm I$. Therefore, in this case we may express our fixed point stratum set as

$$\mathcal{F}^+_I := \big\{(B_1, A_2, B_2, A_3, B_3) \in \mathrm{SU}(2)^5 \ : \ \big[A_2, B_2\big] = \big[A_3, B_3\big] = \pm I\big\}$$

With this form of our fixed point stratum set we may further decompose it into the sub-case where our $A_2, B_2$ and $A_3, B_3$ commutators equal the identity and the sub-case where they equal minus the identity, that is

$$\mathcal{F}^{++}_I := \big\{(B_1, A_2, B_2, A_3, B_3) \in \mathrm{SU}(2)^5 \ : \ \big[A_2, B_2\big] = \big[A_3, B_3\big] = I\big\}$$

and

$$\mathcal{F}^{+-}_I := \big\{(B_1, A_2, B_2, A_3, B_3) \in \mathrm{SU}(2)^5 \ : \ \big[A_2, B_2\big] = \big[A_3, B_3\big] = -I\big\}$$

In this first sub-case of $\mathcal{F}^{++}_I$, we observe that the relations

$$\big[A_2, B_2\big] = I \quad \text{and} \quad \big[A_3, B_3\big] = I$$

implies that $A_2, B_2$ commute and $A_3, B_3$ commute. Recall that any two elements of SU(2) commute if and only if they lie in the same maximal torus, which in SU(2) is conjugate to the subgroup of diagonal matrices. Let T denote a maximal torus in SU(2),

$$\mathrm{T} = \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} : \theta \in [0, 2\pi) \right\}$$

Since $A_2, B_2$ commute, there exists $g \in \mathrm{SU}(2)$ such that

$$g\, A_2 g^{-1}, \ g\, B_2 g^{-1} \in \mathrm{T}$$

This follows from the fact that all maximal tori in SU(2) are conjugate and every element is contained in some maximal torus. Therefore, the set of commuting 2-tuples in SU(2) is

$$\begin{aligned} \mathcal{C}^{++}_I &:= \big\{(a_2, b_2) \in \mathrm{SU}(2)^2 \ : \ [a_2, b_2] = I\big\} \\ &= \big\{(g\, t_1 g^{-1}, g\, t_2 g^{-1}) \ : \ g \in \mathrm{SU}(2), \ t_1, t_2 \in \mathrm{T}\big\} \end{aligned}$$

We wish to show that $\mathcal{C}_I^{++}$ is connected, in order to do so we define the map

$$\Omega : \mathrm{SU}(2) \times \mathrm{T}^2 \longrightarrow \mathcal{C}_I^{++}$$

by

$$\Omega\big(g,(t_1,t_2)\big) = \big(g\,t_1 g^{-1},\ g\,t_2 g^{-1}\big).$$

Observe that the domain of the map, $\mathrm{SU}(2) \times \mathrm{T}^2$, is connected since $\mathrm{SU}(2)$ is connected, T is connected, and the finite product of connected spaces is connected. Therefore, it suffices to show that $\Omega$ is continuous and surjective, as the image of a connected space under a continuous map is connected. We begin by verifying the surjectivity of $\Omega$. Let $(a_2, b_2) \in \mathcal{C}_I^{++}$, then there exists $g \in \mathrm{SU}(2)$ such that

$$g^{-1}a_2 g,\ g^{-1}b_2 g \in \mathrm{T}.$$

This follows from the fact that all maximal tori in SU(2) are conjugate and every element is contained in some maximal torus. Now, if we set

$$t_1 = g^{-1}a_2 g \qquad \text{and} \qquad t_2 = g^{-1}b_2 g.$$

then every element of $\mathcal{C}_I^{++}$ is in the image of $\mathrm{SU}(2) \times \mathrm{T}^2$ under $\Omega$ and so the map is surjective. For the continuity of the map, we note that $\Omega$ is defined on group operations, multiplication and inversion, which are smooth in SU(2), and so the map is continuous. Hence, $\mathcal{C}_I^{++}$ is connected. Note that for this past section about the connectedness of commuting 2-tuples we utilized the commutativity of $A_2$ with $B_2$, however, this argument extends the commutativity of $A_3$ with $B_3$. Now, there are no relations involving $B_1$ in the definition of $\mathcal{F}_I^{++}$, therefore, for any two fixed commuting 2-tuples $\big(A_2, B_2\big)$ and $\big(A_3, B_3\big)$, $B_1$ can be any element in SU(2). Thus,

$$\mathcal{F}_I^{++} \cong \mathcal{C}_I^{++} \times \mathcal{C}_I^{++} \times \mathrm{SU}(2).$$

We know that SU(2) is connected and we have just shown that $\mathcal{C}_I^{++}$ is connected, therefore, since the finite product of connected spaces is connected, it follows that $\mathcal{F}_I^{++}$ is connected. Next let us consider our second sub-case,

$$\mathcal{F}_I^{+-} := \big\{(B_1, A_2, B_2, A_3, B_3) \in \mathrm{SU}(2)^5 \ :\ \big[A_2, B_2\big] = \big[A_3, B_3\big] = -I\big\}$$

In this sub-case, we will take a slightly different but equally valid approach to showing that our fixed point stratum set is connected. To start lets define the commutator map

$$\mathfrak{c} : \mathrm{SU}(2)^2 \longrightarrow \mathrm{SU}(2)$$

given by

$$\big(X, Y\big) \longmapsto \big[X, Y\big]$$

Now observe that our anti-commuting pairs $A_2, B_2$ and $A_3, B_3$ are elements of the fiber over minus the identity of the commutator map, that is

$$\big(A_2, B_2\big), \big(A_3, B_3\big) \in \mathfrak{c}^{-1}(-I) = \big\{(X, Y) \in \mathrm{SU}(2) \ :\ \big[X, Y\big] = -I\big\}$$

Recall that we know that the fibers of this map are connected for SU(2). There are no relations involving $B_1$ in the definition of $\mathcal{F}_I^{+-}$, therefore, for any two fixed anti-commuting 2-tuples $\big(A_2, B_2\big)$ and $\big(A_3, B_3\big)$, $B_1$ can be any element in SU(2). Thus,

$$\mathcal{F}_I^{+-} \cong \mathfrak{c}^{-1}(-I) \times \mathfrak{c}^{-1}(-I) \times \mathrm{SU}(2).$$

Since we know that SU(2) is connected, fiber of the commutator map is connected, and the finite product of connected spaces is connected, it follows that $\mathcal{F}_I^{+-}$ is connected. Note that since $I$ and $-I$ are antipodal points we cannot connect the sets which arise from these two distinct sub-cases, thus in the case where $A_1 = I$ we have two connected components. Now moving on to our second case, where $A_1 = -I$, we have

$$\{-I\} \times \mathcal{F}_{-I}^{+} := \big\{-I\big\} \times \Big\{(B_1, A_2, B_2, A_3, B_3) \in \mathrm{SU}(2)^5 \ :\ \big[A_2, B_2\big] = \big[A_3, B_3\big]^{-1},\ \big(-[A_3, B_3]\big)^2 = I\Big\}$$

Importantly, observe that the proof for this case is nearly identical to that of our previous case, thus we may conclude in a similar fashion that it too has two connected components. □

We have now successfully classified the connectedness of two of our three preimages. Thus we turn our attention to the final preimage which in part partitions our fixed point set. To do so, first we will consider our fixed point stratum sets which arise in the case where the 2-nd power of $A_1$ is in the center of SU(2), specifically when it is equal to minus the identity.

**Lemma 5.** *For every $A_1 \in \mathcal{A}^2_-$, $\{A_1\} \times \mathcal{F}^-_{A_1}$ is connected*

*Proof.* Fix $A_1 \in \mathcal{A}^2_-$. With

$$\{A_1\} \times \mathcal{F}^-_{A_1} := \{A_1\} \times \left\{ (B_1, A_2, B_2, A_3, B_3) \in \mathrm{SU}(2)^5 \, : \, \prod_{i=1}^{3} [A_i, B_i] = I, \; \big([A_3, B_3]A_1\big)^2 = -I \right\}$$

we begin by noting that the singleton set $\{A_1\}$ is connected in SU(2) and the product

$$\{A_1\} \times \mathcal{F}^-_{A_1}$$

is homeomorphic to $\mathcal{F}^-_{A_1}$ as it is just a copy of this set at $A_1$. Thus to determine the connectedness of the fixed point stratum set, it suffices to show the connectedness of $\mathcal{F}^-_{A_1}$. With this, we first define

$$\mathcal{B}^-_{A_1} := \left\{ (B_1, A_3, B_3) \in \mathrm{SU}(2)^4 \, : \, ([A_3, B_3]A_1)^2 = -I \right\}$$

Now consider the map

$$\pi : \mathcal{F}^-_{A_1} \longrightarrow \mathcal{B}^-_{A_1}$$

given by

$$\big(B_1, A_2, B_2, A_3, B_3\big) \longmapsto \big(B_1, A_3, B_3\big)$$

Note that $\pi$ is a restriction of the projection map

$$\mathrm{p} : \mathrm{SU}(2)^5 \longrightarrow \mathrm{SU}(2)^3$$

which we know to be continuous as projections in the product topology are continuous, thus $\pi$ is continuous. Additionally, we claim that $\pi$ is surjective. To see this, take $\big(B_1, A_3, B_3\big) \in \mathcal{B}^-_{A_1}$. Then by the definition of $\mathcal{B}^-_{A_1}$, we have

$$\big([A_3, B_3]A_1\big)^2 = -I$$

Thus our goal is to find an $A_2, B_2 \in \mathrm{SU}(2)$ such that the 5-tuple $\big(B_1, A_2, B_2, A_3, B_3\big)$ lies in $\mathcal{F}^-_{A_1}$, that is

$$\big[A_2, B_2\big] = \big[A_1, B_1\big]^{-1}\big[A_3, B_3\big]^{-1}$$

Recall that every element in SU(2) can be expressed as a commutator. Hence, because

$$\big[A_1, B_1\big]^{-1}\big[A_3, B_3\big]^{-1} \in \mathrm{SU}(2)$$

then by the surjectivity of the commutator map in SU(2), there exists an $X$, $Y \in \mathrm{SU}(2)$ such that

$$\big[X, Y\big] = \big[A_1, B_1\big]^{-1}\big[A_3, B_3\big]^{-1}$$

Define

$$A_2 := X \quad \text{and} \quad B_2 := Y.$$

Now, observe that

$$\begin{aligned} &\big[A_1, B_1\big]\big[X, Y\big]\big[A_3, B_3\big] \\ &= \big[A_1, B_1\big]\Big(\big[A_1, B_1\big]^{-1}\big[A_3, B_3\big]^{-1}\Big)\big[A_3, B_3\big] \\ &= \Big(\big[A_1, B_1\big]\big[A_1, B_1\big]^{-1}\Big)\Big(\big[A_3, B_3\big]^{-1}\big[A_3, B_3\big]\Big) \\ &= I \end{aligned}$$

Hence along with the relation in $\mathcal{B}^-_{A_1}$, it follows that

$$\big(B_1, A_2, B_2, A_3, B_3\big) \in \mathcal{F}^-_{A_1}$$

and thus

$$\pi\big(B_1, A_2, B_2, A_3, B_3\big) = \big(B_1, A_3, B_3\big)$$

so $\pi$ is surjective. Now that we have established that $\pi$ is a continuous surjection, let us consider the fibers of this map. Given a point $\big(B_1, A_3, B_3\big) \in \mathcal{B}^-_{A_1}$, the fiber over this point is defined as

$$\pi^{-1}\big(B_1, A_3, B_3\big) = \big\{(B_1, A_2, B_2, A_3, B_3) \in \mathcal{F}^-_{A_1}\big\}$$

Note that since in $\mathcal{F}^-_{A_1}$ the only relation involving $\big(A_2, B_2\big)$ is the product of the commutators, which we can express as

$$\big[A_2, B_2\big] = \big[A_1, B_1\big]^{-1}\big[A_3, B_3\big]^{-1}$$

we may fix $\big(B_1, A_3, B_3\big) \in \mathcal{B}^-_{A_1}$ and define the map

$$\psi : \pi^{-1}\big(B_1, A_3, B_3\big) \;\longrightarrow\; \big\{(A_2, B_2) \in \mathrm{SU}(2)^2 \,:\, [A_2, B_2] = [A_1, B_1]^{-1}[A_3, B_3]^{-1}\big\}$$

given by

$$\big(B_1, A_2, B_2, A_3, B_3\big) \;\longmapsto\; \big(A_2, B_2\big)$$

Since we fixed our fiber, acting as the domain, by the uniqueness of inverses in SU(2) our map $\psi$ is well-defined and surjective. Moreover, by this uniqueness property and the fact that we fixed our $B_1, A_3$ and $B_3$ arguments, we must have that the map is injective, thus $\psi$ is a bijection. Now we observe that $\psi$ is a restriction of the projection map

$$\mathrm{p} : \mathrm{SU}(2)^5 \;\longrightarrow\; \mathrm{SU}(2)^2$$

which we know to be continuous as projections in the product topology are continuous, thus $\psi$ is continuous. Furthermore, $\psi^{-1}$ is a restriction of the inclusion map

$$\mathfrak{i} : \mathrm{SU}(2)^2 \;\longrightarrow\; \mathrm{SU}(2)^5$$

which we know to be continuous as inclusions in the product topology are continuous, thus $\psi^{-1}$ is continuous. Therefore, $\psi$ is a homeomorphism, that is

$$\pi^{-1}\big(B_1, A_3, B_3\big) \cong \big\{(A_2, B_2) \in \mathrm{SU}(2)^2 \,:\, [A_2, B_2] = [A_1, B_1]^{-1}[A_3, B_3]^{-1}\big\}$$

Importantly, observe that

$$\big\{(A_2, B_2) \in \mathrm{SU}(2)^2 \,:\, [A_2, B_2] = [A_1, B_1]^{-1}[A_3, B_3]^{-1}\big\}$$

is just a fiber over the commutator map of SU(2) and thus is connected. Since we have shown that an arbitrary fiber, $\pi^{-1}(B_1, A_3, B_3)$, is connected, it follows that the fibers of our map $\pi$ are connected. Therefore, $\pi$ is a continuous, surjective map with connected fibers from our total space $\mathcal{F}^-_{A_1}$ into our defined base space $\mathcal{B}^-_{A_1}$. Thus, we will now investigate the connectedness of this base space $\mathcal{B}^-_{A_1}$. To do so consider the 2-nd power map of SU(2)

$$\mathrm{p}_2 : \mathrm{SU}(2) \;\longrightarrow\; \mathrm{SU}(2)$$

given by

$$Z \;\longmapsto\; Z^2$$

Let us examine the fiber over minus the identity

$$\mathrm{p}_2^{-1}(-I) = \big\{Y \in \mathrm{SU}(2) \,:\, (Y)^2 = -I\big\}$$

Note that for any $Y \in \mathrm{p}_2^{-1}(-I)$, $Y$ is a matrix in SU(2), so it is diagonalizable and can be written up to conjugation as

$$Y \sim \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}, \quad \theta \in [0, \pi]$$

Now this represents the conjugacy class determined by the eigenvalues $(e^{i\theta}, e^{-i\theta})$, which by our condition on $Y$ that

$$\big(Y\big)^2 = -I$$

must satisfy

$$\big(e^{i\theta}\big)^2 = -1 \implies e^{i2\theta} = -1 \iff 2\theta \equiv \pi \pmod{2\pi}$$

Therefore, we have

$$\theta = \frac{\pi + 2\pi k}{2} = \frac{(2k+1)\pi}{2}, \quad k \in \mathbb{Z}$$

However, due to the equivalence under conjugation of $\theta \sim -\phi$ and $\theta \sim \theta + 2\pi$ in SU(2), we only consider $\theta \in [0, \pi]$. Thus,

$$0 \leq \frac{(2k+1)\pi}{2} \leq \pi \implies k = 0$$

Hence our only possible eigenvalue pair is

$$\left\{e^{\frac{i\pi}{2}}, e^{\frac{-i\pi}{2}}\right\}$$

This corresponds to a distinct conjugacy class,

$$\mathcal{D}^- := \left\{V \mathrm{diag}\left(e^{\frac{i\pi}{2}}, e^{\frac{-i\pi}{2}}\right) V^{-1} \,:\, V \in \mathrm{SU}(2)\right\}$$

Now, from our relation we know that $[A_3, B_3]A_1$ lies in this fiber of the 2-nd power map, thus

$$[A_3, B_3]A_1 \in \mathcal{D}^-$$

Recall that every conjugacy class of a non-central element in SU(2) is connected and homeomorphic to $S^2$. Therefore we know that

$$\mathcal{D}^- \cong S^2$$

Now note that $B_1$ is not involved in the relation in $\mathcal{B}^-_{A_1}$ and so it is unconstrained in SU(2). We do have, however, that $A_3$ and $B_3$ are involved this relation, thus we will consider the map

$$\Lambda : \mathrm{SU}(2)^2 \longrightarrow \mathrm{SU}(2)$$

given by

$$\big(A_3, B_3\big) \longmapsto \big[A_3, B_3\big]A_1$$

Observe that this is a continuous map as it is defined on a group operation, multiplication, which is smooth in SU(2). The key observation is that by our relation on $\mathcal{B}^-_{A_1}$ we have that

$$\big(B_1, A_3, B_3\big) \in \mathcal{B}^-_{A_1} \iff \Lambda\big(A_3, B_3\big) \in \mathrm{p}_2^{-1}(-I)$$

Therefore,

$$\mathcal{B}^-_{A_1} = \mathrm{SU}(2) \times \Lambda^{-1}\big(\mathrm{p}_2^{-1}(-I)\big)$$

and hence we can express $\mathcal{B}^-_{A_1}$ as

$$\mathcal{B}^-_{A_1} = \mathrm{SU}(2) \times \Lambda^{-1}\big(\mathcal{D}^-\big)$$

Since $\Lambda$ is continuous and the continuous preimage of a connected set is connected, it follows that

$$\Lambda^{-1}\big(\mathcal{D}^-_k\big)$$

is connected. Moreover, since SU(2) is connected and the finite product of a connected spaces is connected, it follows that

$$\mathrm{SU}(2) \times \Lambda^{-1}\big(\mathcal{D}^-\big)$$

is connected. Therefore, our set $\mathcal{B}^-_{A_1}$ is connected. So we have shown that there is a continuous surjection

$$\pi : \mathcal{F}^-_{A_1} \longrightarrow \mathcal{B}^-_{A_1}$$

with non-empty, connected fibers. Since every fiber is non-empty and connected, and the base space $\mathcal{B}^-_{A_1}$ is connected, it follows that $\mathcal{F}^-_{A_1}$ is connected. Hence the entire fixed point stratum set

$$\{A_1\} \times \mathcal{F}^-_{A_1}$$

is connected. □

Having classified the connectedness of each individual fixed point stratum set corresponding to the fibers of $A_1 \in \mathcal{A}^2_-$ under $\mu$, we now wish to consider the union so that we know the number of connected components of the entire preimage.

**Lemma 6.** *For*

$$\mu^{-1}\big(\mathcal{A}^2_-\big) = \bigcup_{A_1 \in \mathcal{A}^2_-} \{A_1\} \times \mathcal{F}^-_{A_1}$$

$\mu^{-1}\big(\mathcal{A}^2_-\big)$ *is connected*

*Proof.* We begin by noting that by the previous lemma, we know that for each $A_1 \in \mathcal{A}^2_-$ our corresponding fixed point stratum set

$$\{A_1\} \times \mathcal{F}^-_{A_1} := \big\{A_1\big\} \times \left\{ (B_1, A_2, B_2, A_3, B_3) \in \mathrm{SU}(2)^5 \,:\, \prod_{i=1}^{3} \big[A_i, B_i\big] = I,\ \big([A_3, B_3]A_1\big)^2 = -I \right\}$$

is connected. Thus the remaining determination of the connectedness of our preiamge

$$\mu^{-1}\big(\mathcal{A}^2_-\big) = \bigcup_{A_1 \in \mathcal{A}^2_-} \{A_1\} \times \mathcal{F}^-_{A_1}$$

depends on that of the parameter space, our characterization set $\mathcal{A}^2_-$. Now with

$$\mathcal{A}^2_- = \big\{A_1 \in \mathrm{SU}(2) \,:\, \big(A_1\big)^2 = -I\big\}$$

let us recall that this is merely the fiber of the identity under the 2-nd power map of SU(2). Thus we will consider this fiber, first defining the 2-nd power map,

$$\mathrm{p}_2 : \mathrm{SU}(2) \longrightarrow \mathrm{SU}(2)$$

given by

$$Z \longmapsto Z^2$$

Let us examine the fiber over minus the identity

$$\mathrm{p}_2^{-1}(-I) = \big\{Y \in \mathrm{SU}(2) \,:\, (Y)^2 = -I\big\}$$

Note that for any $Y \in \mathrm{p}_2^{-1}(-I)$, $Y$ is a matrix in SU(2), so it is diagonalizable and can be written up to conjugation as

$$Y \sim \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}, \quad \theta \in [0, \pi]$$

Now this represents the conjugacy class determined by the eigenvalues $(e^{i\theta}, e^{-i\theta})$, which by our condition on $Y$ that

$$\big(Y\big)^2 = -I$$

must satisfy

$$\big(e^{i\theta}\big)^2 = -1 \implies e^{i2\theta} = -1 \iff 2\theta \equiv \pi \pmod{2\pi}$$

Therefore, we have

$$\theta = \frac{\pi + 2\pi k}{2} = \frac{(2k+1)\pi}{2}, \quad k \in \mathbb{Z}$$

However, due to the equivalence under conjugation of $\theta \sim -\phi$ and $\theta \sim \theta + 2\pi$ in SU(2), we only consider $\theta \in [0, \pi]$. Thus,

$$0 \leq \frac{(2k+1)\pi}{2} \leq \pi \implies k = 0$$

Hence our only possible eigenvalue pair is

$$\left\{ e^{\frac{i\pi}{2}}, e^{\frac{-i\pi}{2}} \right\}$$

This corresponds to a distinct conjugacy class,

$$\mathcal{D}^- := \left\{ V \operatorname{diag}\left( e^{\frac{i\pi}{2}}, e^{\frac{-i\pi}{2}} \right) V^{-1} \, : \, V \in \mathrm{SU}(2) \right\}$$

Now, recall that every conjugacy class of a non-central element in SU(2) is connected and homeomorphic to $S^2$. Therefore we know that

$$\mathcal{D}^- \cong S^2$$

Hence, $\mathcal{A}^2_-$ is connected.

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[INSERT] DR. DUNCAN: Implicit/Inverse Function Theorem Argument

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□

We are now ready to state and prove our main result about the number of connected components of our fixed point set.

**Proposition 2.** *The fixed point set of the 2-nd power of $\Phi^*$, has three connected components.*

*Proof.* With

$$\mathrm{Fix}\big((\Phi^2)^*\big) = \left\{ (A_i, B_i) \in \mathrm{SU}(2)^6 \, : \, \prod_{i=1}^{3} [A_i, B_i] = I, \; \big([A_3, B_3] A_1\big)^2 = \big(A_1\big)^2, \; \big[(A_1)^2, A_3\big] = I, \; \big[(A_1)^2, B_3\big] = I \right\}$$

we begin by recalling that

$$\mathrm{Fix}\big((\Phi^2)^*\big) = \mu^{-1}\big(\mathcal{A}^2_{\not\equiv}\big) \cup \big((\mu^{-1}(I) \cup \mu^{-1}(-I)) \cup \mu^{-1}\big(\mathcal{A}^2_-\big)\big)$$

which is equivalent to

$$\mathrm{Fix}\big((\Phi^2)^*\big) = \bigcup_{A_1 \in \mathcal{A}^2_{\not\equiv}} \{A_1\} \times \mathcal{F}^{\not\equiv}_{A_1} \; \cup \; \{I\} \times \mathcal{F}^+_I \; \cup \; \{-I\} \times \mathcal{F}^+_{-I} \; \cup \bigcup_{A_1'' \in \mathcal{A}^2_-} \{A_1''\} \times \mathcal{F}^-_{A_1''}$$

Now, by a previous lemma we know that the preimage over all $A_1$ whose 2-nd power is non-central in SU(2),

$$\mu^{-1}\big(\mathcal{A}^2_{\not\equiv}\big) = \bigcup_{A_1 \in \mathcal{A}^2_{\not\equiv}} \{A_1\} \times \mathcal{F}^{\not\equiv}_{A_1}$$

has two connected components. By a previous lemma we know that the preimage over all $A_1$ whose 2-nd power is equal to the identity,

$$\mu^{-1}\big(\mathcal{A}^2_+\big) = \{I\} \times \mathcal{F}^+_I \; \cup \; \{-I\} \times \mathcal{F}^+_{-I}$$

has four connected components. Finally, by a previous lemma we know that the preimage over all $A_1$ whose 2-nd power is equal to minus the identity,

$$\mu^{-1}\big(\mathcal{A}^2_-\big) = \bigcup_{A_1'' \in \mathcal{A}^2_-} \{A_1''\} \times \mathcal{F}^-_{A_1''}$$

has one connected component. With this, we note that

$$\bigcup_{A_1 \in \mathcal{A}^2_{\not\pm}} \{A_1\} \times \mathcal{F}^{\not\pm}_{A_1} \,\cup\, \{I\} \times \mathcal{F}^+_I \,\cup\, \{-I\} \times \mathcal{F}^+_{-I} \,\cup \bigcup_{A''_1 \in \mathcal{A}^2_-} \{A''_1\} \times \mathcal{F}^-_{A''_1}$$

is by construction a disjoint union, as necessarily our $A_1$ arguments, which parametrize each individual fixed point stratum set, cannot agree. Therefore, we get an upper bound for the number of connected components of our fixed point set by adding the number of connected components from each respective preimage, that is, we get that our fixed point set has at most seven connected components However, this is just an upper bound and in fact we claim that the actual number of connected components for our fixed point set is much lower. To refine this upper bound on the number of connected components of our fixed point set we will leverage the fact that the closure of a connected set is connected and that if two connected sets intersect in their closure, their union is connected. Thus with this it suffices to show that the intersection of the closures of specific connected components, arising from fixed point stratum sets, are non-empty and thus come together to form larger connected components. First, though, we need to identify which of our connect components could potentially intersect in their closures. To do so let us consider our respective fixed point stratum sets for arbitrary, $A_1 \in \mathcal{A}^2_{\not\pm}$, $A'_1 \in \mathcal{A}^2_+$, and $A''_1 \in \mathcal{A}^2_-$, as these give rise to our connected components. By definition we have

$$\{A_1\} \times \mathcal{F}^{\not\pm}_{A_1} = \{A_1\} \times \Big\{(B_1, A_2, B_2, A_3, B_3) \in \mathrm{SU}(2)^5 \,:\, \big[A_1, B_1\big] = \big[A_2, B_2\big]^{-1},\ \big[A_3, B_3\big] = \big[(A_1)^2, A_3\big] = \big[(A_1)^2, B_3\big] = I$$

$$\{A'_1\} \times \mathcal{F}^+_{A'_1} = \{A'_1\} \times \left\{(B_1, A_2, B_2, A_3, B_3) \in \mathrm{SU}(2)^5 \,:\, \prod_{i=1}^{3} \big[A_i, B_i\big] = I,\ \big([A_3, B_3]A'_1\big)^2 = I\right\}$$

$$\{A''_1\} \times \mathcal{F}^-_{A''_1} = \{A''_1\} \times \left\{(B_1, A_2, B_2, A_3, B_3) \in \mathrm{SU}(2)^5 \,:\, \prod_{i=1}^{3} \big[A_i, B_i\big] = I,\ \big([A_3, B_3]A''_1\big)^2 = -I\right\}$$

Now since SU(2) is a compact Lie group, the closure of a set corresponds to the inclusion of all points that can be approximated by sequences in the set. Since the relations on that set are continuous, any limit point will satisfy those relations as well. Thus if the closures of two subsets of SU(2) intersect, the points in the intersection must satisfy the relations of both sets. With this we observe that for each fixed point stratum set in the union

$$\mu^{-1}\big(\mathcal{A}^2_{\not\pm}\big) = \bigcup_{A_1 \in \mathcal{A}^2_{\not\pm}} \big\{A_1\big\} \times \mathcal{F}^{\not\pm}_{A_1}$$

we have that $A_3$ and $B_3$ commute. Therefore, if we consider the closures of the corresponding two connected components, were the closures of any of the other connected components arising from our other two respective collections of fixed point stratum sets to intersect, they too must satisfy this relation. By the proof of a previous lemma we know that

$$\mu^{-1}\big(\mathcal{A}^2_+\big) = \{I\} \times \mathcal{F}^+_I \,\cup\, \{-I\} \times \mathcal{F}^+_{-I}$$

has two connected components which satisfy this commutation relation. Additionally, for

$$\mu^{-1}\big(\mathcal{A}^2_-\big) = \bigcup_{A''_1 \in \mathcal{A}^2_-} \{A''_1\} \times \mathcal{F}^-_{A''_1}$$

it is connected and there are no restrictions on whether $A_3$ and $B_3$ should be aloud to commute, thus it will satisfy this commutation relation. Note that it immediately follows that all relations for any fixed point stratum set $\{A_1\} \times \mathcal{F}^{\not\pm}_{A_1}$ are satisfied, as by assumption, for each respective fixed point stratum set in the latter preimages, we have by construction that $\big(A'_1\big)^2 = I$ and $\big(A''_1\big)^2 = -I$, thus trivially commute with $A_3$ and $B_3$. It is important to note that these specific fixed point stratum sets decorate the gaps in between the connected components of $\mu^{-1}\big(\mathcal{A}^2_{\not\pm}\big)$ in correspondence to their respective open intervals along $[0, \pi]$ We claim that all of the special connected components of our fixed point stratum sets, that is the ones in which $A_3$ and $B_3$ commute, from our two respective preimages

$$\mu^{-1}\big(\mathcal{A}^2_+\big) = \{I\} \times \mathcal{F}^+_I \ \cup \ \{-I\} \times \mathcal{F}^+_{-I} \quad \text{and} \quad \mu^{-1}\big(\mathcal{A}^2_-\big) = \bigcup_{A''_1 \in \mathcal{A}^2_-} \{A''_1\} \times \mathcal{F}^-_{A''_1}$$

merge all of the connected components from our third preimage, to form one large connected component. This claim may be verified through the previously mentioned intersection of the closure argument. We will look at one case of this a note that we may repeat the argument to sew together all of our target connected components. For this case, fix $A_1 \in \mathrm{SU}(2)$. Then by construction if $\big(A_1\big)^2 = I$ we have that $\mu^{-1}\big(A_1\big) \cong \{A_1\} \times \mathcal{F}^+_{A_1}$. Define the path

$$A_1(t) : \big[0,1\big] \longrightarrow \mu^{-1}\big(\mathrm{SU}(2)\big)$$

where $\big(A_1(0)\big)^2 = I$ and $\big(A_1(t)\big)^2 \neq \pm I$ for $t \neq 0$. Now observe that

$$\lim_{t\to 0} \mu^{-1}\big(A_1(t)\big) \subseteq \{A_1(0)\} \times \mathcal{F}^+_{A_1}$$

Thus the intersection of the closures of the two sets is non-empty and so their union is connected. As we mentioned this is true for all of our special connected components of our fixed point stratum sets, thus we can connect our three special connected components in $\mu^{-1}\big(\mathcal{A}^2_\pm\big)$ and two connected components of $\mu^{-1}\big(\mathcal{A}^2_{\not\pm}\big)$, to form one large connected component. This however is the extent of this merging of connected components that we see from the general fixed point set. This is a result of the following. Suppose that for some fixed $A'_1 \in \mathcal{A}^2_+$, and $A''_1 \in \mathcal{A}^2_-$ we had that

$$\overline{\{A'_1\} \times \mathcal{F}^+_{A'_1}} \ \cap \ \overline{\{A''_1\} \times \mathcal{F}^-_{A''_1}} \neq \emptyset$$

that is the closures of at least one of the connected components from each of the respective connected components from the fixed point stratum sets intersected. Then there would exist an element

$$\big(\tilde{A}_1, \tilde{B}_1, \tilde{A}_2, \tilde{B}_2, \tilde{A}_3, \tilde{B}_3\big) \in \overline{\{A'_1\} \times \mathcal{F}^+_{A'_1}} \ \cap \ \overline{\{A''_1\} \times \mathcal{F}^-_{A''_1}}$$

such that $\tilde{A}_1, \tilde{B}_1, \tilde{A}_2, \tilde{B}_2, \tilde{A}_3$ and $\tilde{B}_3$ satisfy the relations of each respective set, that is

$$\prod_{i=1}^{3} [\tilde{A}_i, \tilde{B}_i] = I$$
$$\big([\tilde{A}_3, \tilde{B}_3]\tilde{A}_1\big)^2 = I$$
$$\big([\tilde{A}_3, \tilde{B}_3]\tilde{A}_1\big)^2 = -I$$

However, in SU(2) the 2-nd power of a matrix cannot simultaneously be equal to both the identity and minus the identity. Therefore, there cannot exists an element

$$\big(\tilde{A}_1, \tilde{B}_1, \tilde{A}_2, \tilde{B}_2, \tilde{A}_3, \tilde{B}_3\big) \in \overline{\{A'_1\} \times \mathcal{F}^+_{A'_1}} \ \cap \ \overline{\{A''_1\} \times \mathcal{F}^-_{A''_1}}$$

and so we are unable to form larger connected components from any of the connected components of each respective fixed point stratum set without the additional components from our other preimage. Therefore the number of connected components in our fixed point set is three and so we are done. □

## 3. Computing Powers of $\Phi$, $(n = 3)$

**3.**

- $T_{\gamma_1} \circ T_{\gamma_2}^{-1}$ :
$$\alpha_1 \mapsto [\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]^{-1} \qquad \beta_1 \mapsto \beta_1[\alpha_3,\beta_3]^{-1}$$
$$\alpha_2 \mapsto \alpha_2 \qquad \beta_2 \mapsto \beta_2$$
$$\alpha_3 \mapsto [\alpha_3,\beta_3]\alpha_1\alpha_3\alpha_1^{-1}[\alpha_3,\beta_3]^{-1} \qquad \beta_3 \mapsto [\alpha_3,\beta_3]\alpha_1\beta_3\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}$$

- $\left(T_{\gamma_1} \circ T_{\gamma_2}^{-1}\right)^2$ :
$$\alpha_1 \mapsto [\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}$$
$$\beta_1 \mapsto \beta_1\alpha_1[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}$$
$$\alpha_2 \mapsto \alpha_2$$
$$\beta_2 \mapsto \beta_2$$
$$\alpha_3 \mapsto [\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1\alpha_3\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}$$
$$\beta_3 \mapsto [\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1\beta_3\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}$$

- $\left(T_{\gamma_1} \circ T_{\gamma_2}^{-1}\right)^3$ :
$$\alpha_1 \mapsto [\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}$$
$$\beta_1 \mapsto \beta_1\alpha_1\alpha_1[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}$$
$$\alpha_2 \mapsto \alpha_2$$
$$\beta_2 \mapsto \beta_2$$
$$\alpha_3 \mapsto [\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1\alpha_3\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}$$
$$\beta_3 \mapsto [\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1\beta_3\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}$$

### 3.1. Fixed Point Equations for $\left((T_{\gamma_1} \circ T_{\gamma_2}^{-1})^3\right)^*$.



- On $R(\Sigma, G)$ :
$$A_1 \mapsto [A_3,B_3]A_1[A_3,B_3]A_1[A_3,B_3]A_1[A_3,B_3]^{-1}A_1^{-1}[A_3,B_3]^{-1}A_1^{-1}[A_3,B_3]^{-1}$$
$$B_1 \mapsto B_1A_1A_1[A_3,B_3]^{-1}A_1^{-1}[A_3,B_3]^{-1}A_1^{-1}[A_3,B_3]^{-1}$$
$$A_2 \mapsto A_2$$
$$B_2 \mapsto B_2$$
$$A_3 \mapsto [A_3,B_3]A_1[A_3,B_3]A_1[A_3,B_3]A_1A_3A_1^{-1}[A_3,B_3]^{-1}A_1^{-1}[A_3,B_3]^{-1}A_1^{-1}[A_3,B_3]^{-1}$$
$$B_3 \mapsto [A_3,B_3]A_1[A_3,B_3]A_1[A_3,B_3]A_1B_3A_1^{-1}[A_3,B_3]^{-1}A_1^{-1}[A_3,B_3]^{-1}A_1^{-1}[A_3,B_3]^{-1}$$

- On $R(\Sigma, G)/G$ :
$$T^{-1}A_1T \mapsto [A_3,B_3]A_1[A_3,B_3]A_1[A_3,B_3]A_1[A_3,B_3]^{-1}A_1^{-1}[A_3,B_3]^{-1}A_1^{-1}[A_3,B_3]^{-1}$$
$$T^{-1}B_1T \mapsto B_1A_1A_1[A_3,B_3]^{-1}A_1^{-1}[A_3,B_3]^{-1}A_1^{-1}[A_3,B_3]^{-1}$$
$$T^{-1}A_2T \mapsto A_2$$
$$T^{-1}B_2T \mapsto B_2$$
$$T^{-1}A_3T \mapsto [A_3,B_3]A_1[A_3,B_3]A_1[A_3,B_3]A_1A_3A_1^{-1}[A_3,B_3]^{-1}A_1^{-1}[A_3,B_3]^{-1}A_1^{-1}[A_3,B_3]^{-1}$$
$$T^{-1}B_3T \mapsto [A_3,B_3]A_1[A_3,B_3]A_1[A_3,B_3]A_1B_3A_1^{-1}[A_3,B_3]^{-1}A_1^{-1}[A_3,B_3]^{-1}A_1^{-1}[A_3,B_3]^{-1}$$

### 3.2. Computing $\Phi^3 = T_{\gamma_1} \circ T_{\gamma_2}^{-1} \circ T_{\gamma_1} \circ T_{\gamma_2}^{-1} \circ T_{\gamma_1} \circ T_{\gamma_2}^{-1}$.

- $\Phi^3(\alpha_1)$ :

$$\begin{aligned}
\Phi^2(\alpha_1) &= [\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1} \\
\Phi\big(\Phi^2(\alpha_1)\big) &= \Phi([\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}) \\
&= \Phi([\alpha_3,\beta_3])\Phi(\alpha_1)\Phi([\alpha_3,\beta_3])\Phi(\alpha_1)\Phi([\alpha_3,\beta_3]^{-1})\Phi(\alpha_1^{-1})\Phi([\alpha_3,\beta_3]^{-1}) \\
&= \Phi([\alpha_3,\beta_3])\Phi(\alpha_1)\Phi([\alpha_3,\beta_3])\Phi(\alpha_1)\big(\Phi([\alpha_3,\beta_3])\big)^{-1}\big(\Phi(\alpha_1)\big)^{-1}\big(\Phi([\alpha_3,\beta_3])\big)^{-1} \\
&= \Phi(\alpha_3\beta_3\alpha_3^{-1}\beta_3^{-1})\Phi(\alpha_1)\Phi(\alpha_3\beta_3\alpha_3^{-1}\beta_3^{-1})\Phi(\alpha_1)\big(\Phi(\alpha_3\beta_3\alpha_3^{-1}\beta_3^{-1})\big)^{-1}\big(\Phi(\alpha_1)\big)^{-1}\big(\Phi(\alpha_3\beta_3\alpha_3^{-1}\beta_3^{-1})\big)^{-1} \\
&= \Phi(\alpha_3)\Phi(\beta_3)\Phi(\alpha_3^{-1})\Phi(\beta_3^{-1})\Phi(\alpha_1)\Phi(\alpha_3)\Phi(\beta_3)\Phi(\alpha_3^{-1})\Phi(\beta_3^{-1})\Phi(\alpha_1)\big(\Phi(\alpha_3)\Phi(\beta_3)\Phi(\alpha_3^{-1})\Phi(\beta_3^{-1})\big)^{-1} \\
&\quad \big(\Phi(\alpha_1)\big)^{-1}\big(\Phi(\alpha_3)\Phi(\beta_3)\Phi(\alpha_3^{-1})\Phi(\beta_3^{-1})\big)^{-1} \\
&= \Phi(\alpha_3)\Phi(\beta_3)\big(\Phi(\alpha_3)\big)^{-1}\big(\Phi(\beta_3)\big)^{-1}\Phi(\alpha_1)\Phi(\alpha_3)\Phi(\beta_3)\big(\Phi(\alpha_3)\big)^{-1}\big(\Phi(\beta_3)\big)^{-1}\Phi(\alpha_1)\big(\Phi(\alpha_3)\Phi(\beta_3) \\
&\quad \big(\Phi(\alpha_3)\big)^{-1}\big(\Phi(\beta_3)\big)^{-1}\big)\big)^{-1}\big(\Phi(\alpha_1)\big)^{-1}\big(\Phi(\alpha_3)\Phi(\beta_3)\big(\Phi(\alpha_3)\big)^{-1}\big(\Phi(\beta_3)\big)^{-1}\big)^{-1} \\
&= [\alpha_3,\beta_3]\alpha_1\alpha_3\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1\beta_3\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1\alpha_3^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1\beta_3^{-1} \\
&\quad \alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1\alpha_3\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1\beta_3\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1\alpha_3^{-1} \\
&\quad \alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1\beta_3^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1\beta_3\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1\alpha_3 \\
&\quad \alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1\beta_3^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1\alpha_3^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1\beta_3 \\
&\quad \alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1\alpha_3\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1\beta_3^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1\alpha_3^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1} \\
&= [\alpha_3,\beta_3]\alpha_1\alpha_3\beta_3\alpha_3^{-1}\beta_3^{-1}\alpha_1\alpha_3\beta_3\alpha_3^{-1}\beta_3^{-1}\alpha_1\beta_3\alpha_3\beta_3^{-1}\alpha_3^{-1}\alpha_1^{-1}\beta_3\alpha_3\beta_3^{-1}\alpha_3^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1} \\
&= [\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1[\beta_3,\alpha_3]\alpha_1^{-1}[\beta_3,\alpha_3]\alpha_1^{-1}[\alpha_3,\beta_3]^{-1} \\
&= [\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}
\end{aligned}$$

$$\boxed{\Phi^3(\alpha_1) = [\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}}$$

- $\Phi^3(\beta_1)$ :

$$\begin{aligned}
\Phi^2(\beta_1) &= \beta_1\alpha_1[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1} \\
\Phi\big(\Phi^2(\beta_1)\big) &= \Phi(\beta_1\alpha_1[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}) \\
&= \Phi(\beta_1)\Phi(\alpha_1)\Phi([\alpha_3,\beta_3]^{-1})\Phi(\alpha_1^{-1})\Phi([\alpha_3,\beta_3]^{-1}) \\
&= \Phi(\beta_1)\Phi(\alpha_1)\big(\Phi([\alpha_3,\beta_3])\big)^{-1}\big(\Phi(\alpha_1)\big)^{-1}\big(\Phi([\alpha_3,\beta_3])\big)^{-1} \\
&= \Phi(\beta_1)\Phi(\alpha_1)\big(\Phi(\alpha_3\beta_3\alpha_3^{-1}\beta_3^{-1})\big)^{-1}\big(\Phi(\alpha_1)\big)^{-1}\big(\Phi(\alpha_3\beta_3\alpha_3^{-1}\beta_3^{-1})\big)^{-1} \\
&= \Phi(\beta_1)\Phi(\alpha_1)\big(\Phi(\alpha_3)\Phi(\beta_3)\Phi(\alpha_3^{-1})\Phi(\beta_3^{-1})\big)^{-1}\big(\Phi(\alpha_1)\big)^{-1}\big(\Phi(\alpha_3)\Phi(\beta_3)\Phi(\alpha_3^{-1})\Phi(\beta_3^{-1})\big)^{-1} \\
&= \Phi(\beta_1)\Phi(\alpha_1)\big(\Phi(\alpha_3)\Phi(\beta_3)\big(\Phi(\alpha_3)\big)^{-1}\big(\Phi(\beta_3)\big)^{-1}\big)^{-1}\big(\Phi(\alpha_1)\big)^{-1}\big(\Phi(\alpha_3)\Phi(\beta_3)\big(\Phi(\alpha_3)\big)^{-1}\big(\Phi(\beta_3)\big)^{-1}\big)^{-1} \\
&= \beta_1[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1\beta_3\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1\alpha_3\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1\beta_3^{-1} \\
&\quad \alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1\alpha_3^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1\beta_3\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1\alpha_3 \\
&\quad \alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1\beta_3^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1\alpha_3^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1} \\
&= \beta_1\alpha_1\alpha_1\beta_3\alpha_3\beta_3^{-1}\alpha_3^{-1}\alpha_1^{-1}\beta_3\alpha_3\beta_3^{-1}\alpha_3^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1} \\
&= \beta_1\alpha_1\alpha_1[\beta_3,\alpha_3]\alpha_1^{-1}[\beta_3,\alpha_3]\alpha_1^{-1}[\alpha_3,\beta_3]^{-1} \\
&= \beta_1\alpha_1\alpha_1[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}
\end{aligned}$$

$$\boxed{\Phi^3(\beta_1) = \beta_1\alpha_1\alpha_1[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}}$$

- $\Phi^3(\alpha_2)$ :

$$\begin{aligned}\Phi^2(\alpha_2) &= \alpha_2\\ \Phi\big(\Phi^2(\alpha_2)\big) &= \Phi(\alpha_2)\\ &= \alpha_2\end{aligned}$$

$$\boxed{\Phi^3(\alpha_2) = \alpha_2}$$

- $\Phi^3(\beta_2)$ :

$$\begin{aligned}\Phi^2(\beta_2) &= \beta_2\\ \Phi\big(\Phi^2(\beta_2)\big) &= \Phi(\beta_2)\\ &= \beta_2\end{aligned}$$

$$\boxed{\Phi^3(\beta_2) = \beta_2}$$

- $\Phi^3(\alpha_3)$ :

$$\begin{aligned}
\Phi^2(\alpha_3) &= [\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1\alpha_3\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\\
\Phi\big(\Phi^2(\alpha_3)\big) &= \Phi([\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1\alpha_3\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1})\\
&= \Phi([\alpha_3,\beta_3])\Phi(\alpha_1)\Phi([\alpha_3,\beta_3])\Phi(\alpha_1)\Phi(\alpha_3)\Phi(\alpha_1^{-1})\Phi([\alpha_3,\beta_3]^{-1})\Phi(\alpha_1^{-1})\Phi([\alpha_3,\beta_3]^{-1})\\
&= \Phi([\alpha_3,\beta_3])\Phi(\alpha_1)\Phi([\alpha_3,\beta_3])\Phi(\alpha_1)\Phi(\alpha_3)\big(\Phi(\alpha_1)\big)^{-1}\big(\Phi([\alpha_3,\beta_3])\big)^{-1}\big(\Phi(\alpha_1)\big)^{-1}\big(\Phi([\alpha_3,\beta_3])\big)^{-1}\\
&= \Phi(\alpha_3\beta_3\alpha_3^{-1}\beta_3^{-1})\Phi(\alpha_1)\Phi(\alpha_3\beta_3\alpha_3^{-1}\beta_3^{-1})\Phi(\alpha_1)\Phi(\alpha_3)\big(\Phi(\alpha_1)\big)^{-1}\big(\Phi(\alpha_3\beta_3\alpha_3^{-1}\beta_3^{-1})\big)^{-1}\\
&\quad\big(\Phi(\alpha_1)\big)^{-1}\big(\Phi(\alpha_3\beta_3\alpha_3^{-1}\beta_3^{-1})\big)^{-1}\\
&= \Phi(\alpha_3)\Phi(\beta_3)\Phi(\alpha_3^{-1})\Phi(\beta_3^{-1})\Phi(\alpha_1)\Phi(\alpha_3)\Phi(\beta_3)\Phi(\alpha_3^{-1})\Phi(\beta_3^{-1})\Phi(\alpha_1)\Phi(\alpha_3)\big(\Phi(\alpha_1)\big)^{-1}\\
&\quad\big(\Phi(\alpha_3)\Phi(\beta_3)\Phi(\alpha_3^{-1})\Phi(\beta_3^{-1})\big)^{-1}\big(\Phi(\alpha_1)\big)^{-1}\big(\Phi(\alpha_3)\Phi(\beta_3)\Phi(\alpha_3^{-1})\Phi(\beta_3^{-1})\big)^{-1}\\
&= \Phi(\alpha_3)\Phi(\beta_3)\big(\Phi(\alpha_3)\big)^{-1}\big(\Phi(\beta_3)\big)^{-1}\Phi(\alpha_1)\Phi(\alpha_3)\Phi(\beta_3)\big(\Phi(\alpha_3)\big)^{-1}\big(\Phi(\beta_3)\big)^{-1}\Phi(\alpha_1)\Phi(\alpha_3)\big(\Phi(\alpha_1)\big)^{-1}\\
&\quad\Big(\Phi(\alpha_3)\Phi(\beta_3)\big(\Phi(\alpha_3)\big)^{-1}\big(\Phi(\beta_3)\big)^{-1}\Big)^{-1}\big(\Phi(\alpha_1)\big)^{-1}\Big(\Phi(\alpha_3)\Phi(\beta_3)\big(\Phi(\alpha_3)\big)^{-1}\big(\Phi(\beta_3)\big)^{-1}\Big)^{-1}\\
&= [\alpha_3,\beta_3]\alpha_1\alpha_3\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1\beta_3\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1\alpha_3^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1\beta_3^{-1}\\
&\quad\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1\alpha_3\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1\beta_3\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1\alpha_3^{-1}\\
&\quad\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1\beta_3^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1\alpha_3\alpha_1^{-1}\\
&\quad[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1\beta_3\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1\alpha_3\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1\beta_3^{-1}\\
&\quad\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1\alpha_3^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1\beta_3\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1\alpha_3\\
&\quad\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1\beta_3^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1\alpha_3^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\\
&= [\alpha_3,\beta_3]\alpha_1\alpha_3\beta_3\alpha_3^{-1}\beta_3^{-1}\alpha_1\alpha_3\beta_3\alpha_3^{-1}\beta_3^{-1}\alpha_1\alpha_3\alpha_1^{-1}\beta_3\alpha_3\beta_3^{-1}\alpha_3^{-1}\alpha_1^{-1}\beta_3\alpha_3\beta_3^{-1}\alpha_3^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\\
&= [\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1\alpha_3\alpha_1^{-1}[\beta_3,\alpha_3]\alpha_1^{-1}[\beta_3,\alpha_3]\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\\
&= [\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1\alpha_3\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}
\end{aligned}$$

$$\boxed{\Phi^3(\alpha_3) = [\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1\alpha_3\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}}$$

- $\Phi^3(\beta_3)$ :

$$
\begin{aligned}
\Phi^2(\beta_3) &= [\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1\beta_3\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1} \\
\Phi\big(\Phi^2(\alpha_3)\big) &= \Phi([\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1\beta_3\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}) \\
&= \Phi([\alpha_3,\beta_3])\Phi(\alpha_1)\Phi([\alpha_3,\beta_3])\Phi(\alpha_1)\Phi(\beta_3)\Phi(\alpha_1^{-1})\Phi([\alpha_3,\beta_3]^{-1})\Phi(\alpha_1^{-1})\Phi([\alpha_3,\beta_3]^{-1}) \\
&= \Phi([\alpha_3,\beta_3])\Phi(\alpha_1)\Phi([\alpha_3,\beta_3])\Phi(\alpha_1)\Phi(\beta_3)\big(\Phi(\alpha_1)\big)^{-1}\big(\Phi([\alpha_3,\beta_3])\big)^{-1}\big(\Phi(\alpha_1)\big)^{-1}\big(\Phi([\alpha_3,\beta_3])\big)^{-1} \\
&= \Phi(\alpha_3\beta_3\alpha_3^{-1}\beta_3^{-1})\Phi(\alpha_1)\Phi(\alpha_3\beta_3\alpha_3^{-1}\beta_3^{-1})\Phi(\alpha_1)\Phi(\beta_3)\big(\Phi(\alpha_1)\big)^{-1}\big(\Phi(\alpha_3\beta_3\alpha_3^{-1}\beta_3^{-1})\big)^{-1} \\
&\quad \big(\Phi(\alpha_1)\big)^{-1}\big(\Phi(\alpha_3\beta_3\alpha_3^{-1}\beta_3^{-1})\big)^{-1} \\
&= \Phi(\alpha_3)\Phi(\beta_3)\Phi(\alpha_3^{-1})\Phi(\beta_3^{-1})\Phi(\alpha_1)\Phi(\alpha_3)\Phi(\beta_3)\Phi(\alpha_3^{-1})\Phi(\beta_3^{-1})\Phi(\alpha_1)\Phi(\beta_3)\big(\Phi(\alpha_1)\big)^{-1} \\
&\quad \big(\Phi(\alpha_3)\Phi(\beta_3)\Phi(\alpha_3^{-1})\Phi(\beta_3^{-1})\big)^{-1}\big(\Phi(\alpha_1)\big)^{-1}\big(\Phi(\alpha_3)\Phi(\beta_3)\Phi(\alpha_3^{-1})\Phi(\beta_3^{-1})\big)^{-1} \\
&= \Phi(\alpha_3)\Phi(\beta_3)\big(\Phi(\alpha_3)\big)^{-1}\big(\Phi(\beta_3)\big)^{-1}\Phi(\alpha_1)\Phi(\alpha_3)\Phi(\beta_3)\big(\Phi(\alpha_3)\big)^{-1}\big(\Phi(\beta_3)\big)^{-1}\Phi(\alpha_1)\Phi(\beta_3)\big(\Phi(\alpha_1)\big)^{-1} \\
&\quad \Big(\Phi(\alpha_3)\Phi(\beta_3)\big(\Phi(\alpha_3)\big)^{-1}\big(\Phi(\beta_3)\big)^{-1}\Big)^{-1}\big(\Phi(\alpha_1)\big)^{-1}\Big(\Phi(\alpha_3)\Phi(\beta_3)\big(\Phi(\alpha_3)\big)^{-1}\big(\Phi(\beta_3)\big)^{-1}\Big)^{-1} \\
&= [\alpha_3,\beta_3]\alpha_1\alpha_3\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1\beta_3\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1\alpha_3^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1\beta_3^{-1} \\
&\quad \alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1\alpha_3\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1\beta_3\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1\alpha_3^{-1} \\
&\quad \alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1\beta_3^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1\beta_3\alpha_1^{-1} \\
&\quad [\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1\beta_3\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1\alpha_3\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1\beta_3^{-1} \\
&\quad \alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1\alpha_3^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1\beta_3\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1\alpha_3 \\
&\quad \alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1\beta_3^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1\alpha_3^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1} \\
&= [\alpha_3,\beta_3]\alpha_1\alpha_3\beta_3\alpha_3^{-1}\beta_3^{-1}\alpha_1\alpha_3\beta_3\alpha_3^{-1}\beta_3^{-1}\alpha_1\beta_3\alpha_1^{-1}\beta_3\alpha_3\beta_3^{-1}\alpha_3^{-1}\alpha_1^{-1}\beta_3\alpha_3\beta_3^{-1}\alpha_3^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1} \\
&= [\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1\beta_3\alpha_1^{-1}[\beta_3,\alpha_3]\alpha_1^{-1}[\beta_3,\alpha_3]\alpha_1^{-1}[\alpha_3,\beta_3]^{-1} \\
&= [\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1\beta_3\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}
\end{aligned}
$$

$$
\boxed{\Phi^3(\beta_3) = [\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1\beta_3\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}}
$$

3.3. **Checking Relation.**

- $\Phi^3 := T_{\gamma_1} \circ T_{\gamma_2}^{-1} \circ T_{\gamma_1} \circ T_{\gamma_2}^{-1} \circ T_{\gamma_1} \circ T_{\gamma_2}^{-1}$ :

(i) $\left[\Phi^3(\alpha_1), \Phi^3(\beta_1)\right]\left[\Phi^3(\alpha_2), \Phi^3(\beta_2)\right]\left[\Phi^3(\alpha_1), \Phi^3(\beta_1)\right]$

(ii) $\left[[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}, \beta_1\alpha_1\alpha_1[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\right]$
$\left[\alpha_2,\beta_2\right][\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1\left[\alpha_3,\beta_3\right]\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}$

(iii) $[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\beta_1\alpha_1\alpha_1[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}$
$[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1^{-1}\alpha_1^{-1}\beta_1^{-1}$
$\left[\alpha_2,\beta_2\right]\left[\alpha_3,\beta_3\right]\alpha_1[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}$

(iv) $[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}\alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1}$
$\left[\alpha_2,\beta_2\right]\left[\alpha_3,\beta_3\right]\alpha_1[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}$

(v) $[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}\alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1}$
$\left[\alpha_2,\beta_2\right]\left[\alpha_3,\beta_3\right]\alpha_1[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}$

(vi) $[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}\left[\alpha_1,\beta_1\right]$
$\left[\alpha_2,\beta_2\right]\left[\alpha_3,\beta_3\right]\alpha_1[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}$

(vii) $[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}$
$\alpha_1[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}$

(viii) $\boxed{1}$

## 4. General Powers of Φ

**4.**

- $T_{\gamma_1} \circ T_{\gamma_2}^{-1}$ :
  $\alpha_1 \mapsto [\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]^{-1}$
  $\alpha_2 \mapsto \alpha_2$
  $\alpha_3 \mapsto [\alpha_3,\beta_3]\alpha_1\alpha_3\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}$
  $\beta_1 \mapsto \beta_1[\alpha_3,\beta_3]^{-1}$
  $\beta_2 \mapsto \beta_2$
  $\beta_3 \mapsto [\alpha_3,\beta_3]\alpha_1\beta_3\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}$

- $\left(T_{\gamma_1} \circ T_{\gamma_2}^{-1}\right)^2$ :
  $\alpha_1 \mapsto [\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}$
  $\beta_1 \mapsto \beta_1\alpha_1[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}$
  $\alpha_2 \mapsto \alpha_2$
  $\beta_2 \mapsto \beta_2$
  $\alpha_3 \mapsto [\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1\alpha_3\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}$
  $\beta_3 \mapsto [\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1\beta_3\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}$

- $\left(T_{\gamma_1} \circ T_{\gamma_2}^{-1}\right)^3$ :
  $\alpha_1 \mapsto [\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}$
  $\beta_1 \mapsto \beta_1\alpha_1\alpha_1[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}$
  $\alpha_2 \mapsto \alpha_2$
  $\beta_2 \mapsto \beta_2$
  $\alpha_3 \mapsto [\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1\alpha_3\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}$
  $\beta_3 \mapsto [\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]\alpha_1\beta_3\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}$

- $\big(T_{\gamma_1}\circ T_{\gamma_2}^{-1}\big)^n:$ $$\begin{aligned} \alpha_1 &\mapsto \big([\alpha_3,\beta_3]\alpha_1\big)^n\alpha_1\big(\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\big)^n\\ \beta_1 &\mapsto \beta_1\big(\alpha_1\big)^n\big(\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\big)^n\\ \alpha_2 &\mapsto \alpha_2\\ \beta_2 &\mapsto \beta_2\\ \alpha_3 &\mapsto \big([\alpha_3,\beta_3]\alpha_1\big)^n\,\alpha_3\,\big(\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\big)^n\\ \beta_3 &\mapsto \big([\alpha_3,\beta_3]\alpha_1\big)^n\,\beta_3\,\big(\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\big)^n \end{aligned}$$

### 4.1. Fixed Point Equations for $\big((T_{\gamma_1}\circ T_{\gamma_2}^{-1})^n\big)^*$.



- On $R(\Sigma,G):$ $$\begin{aligned} A_1 &\mapsto \big([A_3,B_3]A_1\big)^nA_1\big(A_1^{-1}[A_3,B_3]^{-1}\big)^n\\ B_1 &\mapsto B_1\big(A_1\big)^n\big(A_1^{-1}[A_3,B_3]^{-1}\big)^n\\ A_2 &\mapsto A_2\\ B_2 &\mapsto B_2\\ A_3 &\mapsto \big([A_3,B_3]A_1\big)^n\,A_3\,\big(A_1^{-1}[A_3,B_3]^{-1}\big)^n\\ B_3 &\mapsto \big([A_3,B_3]A_1\big)^n\,B_3\,\big(A_1^{-1}[A_3,B_3]^{-1}\big)^n \end{aligned}$$
- On $R(\Sigma,G)/G:$ $$\begin{aligned} T^{-1}A_1T &\mapsto \big([A_3,B_3]A_1\big)^nA_1\big(A_1^{-1}[A_3,B_3]^{-1}\big)^n\\ T^{-1}B_1T &\mapsto B_1\big(A_1\big)^n\big(A_1^{-1}[A_3,B_3]^{-1}\big)^n\\ T^{-1}A_2T &\mapsto A_2\\ T^{-1}B_2T &\mapsto B_2\\ T^{-1}A_3T &\mapsto \big([A_3,B_3]A_1\big)^n\,A_3\,\big(A_1^{-1}[A_3,B_3]^{-1}\big)^n\\ T^{-1}B_3T &\mapsto \big([A_3,B_3]A_1\big)^n\,B_3\,\big(A_1^{-1}[A_3,B_3]^{-1}\big)^n \end{aligned}$$

### 4.2. Proving General $\Phi^n$ Equations.



**Lemma 7.** *$\Phi\big(([\alpha_3,\beta_3]\alpha_1)^n\big)=\big([\alpha_3,\beta_3]\alpha_1\big)^n$ for every $n\in\mathbb{Z}\setminus\{0\}$*

*Proof.* Fix $n\in\mathbb{Z}\setminus\{0\}$. We begin by noting that since $\Phi$ is an automorphism it follows that

$$\begin{aligned}\Phi\big(([\alpha_3,\beta_3]\alpha_1)^n\big) &= \big(\Phi([\alpha_3,\beta_3]\alpha_1)\big)^n\\ &= \big(\Phi([\alpha_3,\beta_3])\Phi(\alpha_1)\big)^n\end{aligned}$$

and thus it suffices to show that

$$\Phi([\alpha_3,\beta_3])\Phi(\alpha_1)=[\alpha_3,\beta_3]\alpha_1$$

We will proceed with manual computations.

$$\begin{aligned}\Phi([\alpha_3,\beta_3])\Phi(\alpha_1) &= \Phi(\alpha_3\beta_3\alpha_3^{-1}\beta_3^{-1})\Phi(\alpha_1)\\ &= \Phi(\alpha_3)\Phi(\beta_3)\Phi(\alpha_3^{-1})\Phi(\beta_3^{-1})\Phi(\alpha_1)\\ &= \Phi(\alpha_3)\Phi(\beta_3)\big(\Phi(\alpha_3)\big)^{-1}\big(\Phi(\beta_3)\big)^{-1}\Phi(\alpha_1)\\ &= [\alpha_3,\beta_3]\alpha_1\alpha_3\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1\beta_3\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1\alpha_3^{-1}\\ &\quad\ \alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1\beta_3^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]^{-1}\\ &= [\alpha_3,\beta_3]\alpha_1\alpha_3\beta_3\alpha_3^{-1}\beta_3^{-1}[\alpha_3,\beta_3]^{-1}\\ &= [\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3][\alpha_3,\beta_3]^{-1}\\ &= [\alpha_3,\beta_3]\alpha_1\end{aligned}$$

Therefore

$$\Phi([\alpha_3,\beta_3])\Phi(\alpha_1) = [\alpha_3,\beta_3]\alpha_1$$

and so we are done. □

**Lemma 8.** $\Phi\big((\alpha_1^{-1}[\alpha_3,\beta_3]^{-1})^n\big) = \big(\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\big)^n$ *for every* $n \in \mathbb{Z}\setminus\{0\}$

*Proof.* Fix $n \in \mathbb{Z}\setminus\{0\}$. We begin by noting that since $\Phi$ is an automorphism it follows that

$$\begin{aligned}\Phi\big((\alpha_1^{-1}[\alpha_3,\beta_3]^{-1})^n\big) &= \big(\Phi(\alpha_1^{-1}[\alpha_3,\beta_3]^{-1})\big)^n\\ &= \big(\Phi(\alpha_1^{-1})\Phi([\alpha_3,\beta_3]^{-1})\big)^n\end{aligned}$$

and thus it suffices to show that

$$\Phi(\alpha_1^{-1})\Phi([\alpha_3,\beta_3]^{-1}) = \alpha_1^{-1}[\alpha_3,\beta_3]^{-1}$$

We will proceed with manual computations.

$$\begin{aligned}\Phi(\alpha_1^{-1})\Phi([\alpha_3,\beta_3]^{-1}) &= \Phi(\alpha_1^{-1})\Phi(\beta_3\alpha_3\beta_3^{-1}\alpha_3^{-1})\\ &= \Phi(\alpha_1^{-1})\Phi(\beta_3)\Phi(\alpha_3)\Phi(\beta_3^{-1})\Phi(\alpha_3^{-1})\\ &= \big(\Phi(\alpha_1)\big)^{-1}\Phi(\beta_3)\Phi(\alpha_3)\big(\Phi(\beta_3)\big)^{-1}\big(\Phi(\alpha_3)\big)^{-1}\\ &= [\alpha_3,\beta_3]\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1\beta_3\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1\alpha_3\\ &\quad\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1\beta_3^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3]\alpha_1\alpha_3^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\\ &= [\alpha_3,\beta_3]\beta_3\alpha_3\beta_3^{-1}\alpha_3^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\\ &= [\alpha_3,\beta_3][\beta_3,\alpha_3]\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\\ &= [\alpha_3,\beta_3][\alpha_3,\beta_3]^{-1}\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\\ &= \alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\end{aligned}$$

Therefore

$$\Phi(\alpha_1^{-1})\Phi([\alpha_3,\beta_3]^{-1}) = \alpha_1^{-1}[\alpha_3,\beta_3]^{-1}$$

and so we are done. □

Now we will prove that our general equations hold:

- $\Phi^n(\alpha_1) = \big([\alpha_3,\beta_3]\alpha_1\big)^n\alpha_1\big(\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\big)^n$

  *Proof.* We will proceed by induction. First we note that for $n = 1$ we have

$$\big([\alpha_3,\beta_3]\alpha_1\big)\alpha_1\big(\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\big) = [\alpha_3,\beta_3]\alpha_1\alpha_1\alpha_1^{-1}[\alpha_3,\beta_3]^{-1} = [\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]^{-1}$$

  which is our $\Phi(\alpha_1)$ and thus our base case holds. Next, suppose that

$$\Phi^n(\alpha_1) = \big([\alpha_3,\beta_3]\alpha_1\big)^n\alpha_1\big(\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\big)^n$$

  for some $n \in \mathbb{Z}_{>0}$. We wish to show that for $n+1$

$$\Phi^{n+1}(\alpha_1) = \big([\alpha_3,\beta_3]\alpha_1\big)^{n+1}\alpha_1\big(\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\big)^{n+1}$$

  We begin by noting that

$$\Phi^{n+1}(\alpha_1) = \Phi\big(\Phi^n(\alpha_1)\big)$$

  thus by our induction hypothesis we have

$$\begin{aligned}\Phi\big(\Phi^n(\alpha_1)\big) &= \Phi\left(([\alpha_3,\beta_3]\alpha_1)^n\alpha_1(\alpha_1^{-1}[\alpha_3,\beta_3]^{-1})^n\right)\\ &= \Phi\big(([\alpha_3,\beta_3]\alpha_1)^n\big)\Phi(\alpha_1)\Phi\big((\alpha_1^{-1}[\alpha_3,\beta_3]^{-1})^n\big)\end{aligned}$$

Now by our previous lemmas this becomes

$$\begin{aligned}\Phi^{n+1}(\alpha_1) &= \Phi\big(([\alpha_3,\beta_3]\alpha_1)^n\big)\Phi(\alpha_1)\Phi\big((\alpha_1^{-1}[\alpha_3,\beta_3]^{-1})^n\big)\\ &= \big([\alpha_3,\beta_3]\alpha_1\big)^n[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]^{-1}\big(\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\big)^n\\ &= \big([\alpha_3,\beta_3]\alpha_1\big)^{n+1}\alpha_1\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\big(\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\big)^n\\ &= \big([\alpha_3,\beta_3]\alpha_1\big)^{n+1}\alpha_1\big(\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\big)^{n+1}\end{aligned}$$

and so we are done. □

- $\Phi^n(\beta_1) = \beta_1\big(\alpha_1\big)^n\big(\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\big)^n$

*Proof.* We will proceed by induction. First we note that for $n = 1$ we have

$$\beta_1\big(\alpha_1\big)\big(\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\big) = \beta_1\alpha_1\alpha_1^{-1}[\alpha_3,\beta_3]^{-1} = \beta_3[\alpha_3,\beta_3]^{-1}$$

which is our $\Phi(\alpha_1)$ and thus our base case holds. Next, suppose that

$$\Phi^n(\beta_1) = \beta_1\big(\alpha_1\big)^n\big(\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\big)^n$$

for some $n \in \mathbb{Z}_{>0}$. We wish to show that for $n+1$

$$\Phi^{n+1}(\beta_1) = \beta_1\big(\alpha_1\big)^{n+1}\big(\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\big)^{n+1}$$

We begin by noting that

$$\Phi^{n+1}(\beta_1) = \Phi\big(\Phi^n(\beta_1)\big)$$

thus by our induction hypothesis we have

$$\begin{aligned}\Phi\big(\Phi^n(\beta_1)\big) &= \Phi\left(\beta_1(\alpha_1)^n(\alpha_1^{-1}[\alpha_3,\beta_3]^{-1})^n\right)\\ &= \Phi(\beta_1)\Phi((\alpha_1)^n)\Phi((\alpha_1^{-1}[\alpha_3,\beta_3]^{-1})^n)\\ &= \Phi(\beta_1)\big(\Phi(\alpha_1)\big)^n\Phi((\alpha_1^{-1}[\alpha_3,\beta_3]^{-1})^n)\\ &= \beta_1[\alpha_3,\beta_3]^{-1}([\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]^{-1})^n(\alpha_1^{-1}[\alpha_3,\beta_3]^{-1})^n\\ &= \beta_1[\alpha_3,\beta_3]^{-1}[\alpha_3,\beta_3](\alpha_1)^n\alpha_1\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}(\alpha_1^{-1}[\alpha_3,\beta_3]^{-1})^n\\ &= \beta_1(\alpha_1)^{n+1}(\alpha_1^{-1}[\alpha_3,\beta_3]^{-1})^{n+1}\end{aligned}$$

and so we are done. □

- $\Phi^n(\alpha_2) = \alpha_2$

*Proof.* We note that under $\Phi$ our $\alpha_2$ is fixed, thus for any power of $\Phi$ it remains fixed. □

- $\Phi^n(\beta_2) = \beta_2$

*Proof.* We note that under $\Phi$ our $\beta_2$ is fixed, thus for any power of $\Phi$ it remains fixed. □

- $\Phi^n(\alpha_3) = \big([\alpha_3,\beta_3]\alpha_1\big)^n\alpha_3\big(\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\big)^n$

*Proof.* We will proceed by induction. First we note that for $n = 1$ we have

$$\big([\alpha_3,\beta_3]\alpha_1\big)\alpha_3\big(\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\big) = [\alpha_3,\beta_3]\alpha_1\alpha_3\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}$$

which is our $\Phi(\alpha_3)$ and thus our base case holds. Next, suppose that

$$\Phi^n(\alpha_3) = \big([\alpha_3,\beta_3]\alpha_1\big)^n\alpha_3\big(\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\big)^n$$

for some $n \in \mathbb{Z}_{>0}$. We wish to show that for $n+1$

$$\Phi^{n+1}(\alpha_3) = \big([\alpha_3,\beta_3]\alpha_1\big)^{n+1}\alpha_3\big(\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\big)^{n+1}$$

We begin by noting that

$$\Phi^{n+1}(\alpha_3) = \Phi\big(\Phi^n(\alpha_3)\big)$$

thus by our induction hypothesis we have

$$\begin{aligned}\Phi\big(\Phi^n(\alpha_3)\big) &= \Phi\left(([\alpha_3,\beta_3]\alpha_1)^n\alpha_3(\alpha_1^{-1}[\alpha_3,\beta_3]^{-1})^n\right)\\ &= \Phi\big(([\alpha_3,\beta_3]\alpha_1)^n\big)\Phi(\alpha_3)\Phi\big((\alpha_1^{-1}[\alpha_3,\beta_3]^{-1})^n\big)\end{aligned}$$

Now by our previous lemmas this becomes

$$\begin{aligned}\Phi^{n+1}(\alpha_3) &= \Phi\big(([\alpha_3,\beta_3]\alpha_1)^n\big)\Phi(\alpha_3)\Phi\big((\alpha_1^{-1}[\alpha_3,\beta_3]^{-1})^n\big)\\ &= \big([\alpha_3,\beta_3]\alpha_1\big)^n[\alpha_3,\beta_3]\alpha_1\alpha_3\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\big(\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\big)^n\\ &= \big([\alpha_3,\beta_3]\alpha_1\big)^{n+1}\alpha_3\big(\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\big)^{n+1}\end{aligned}$$

and so we are done. □

- $\Phi^n(\beta_3) = \big([\alpha_3,\beta_3]\alpha_1\big)^n\beta_3\big(\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\big)^n$

*Proof.* We will proceed by induction. First we note that for $n = 1$ we have

$$\big([\alpha_3,\beta_3]\alpha_1\big)\beta_3\big(\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\big) = [\alpha_3,\beta_3]\alpha_1\beta_3\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}$$

which is our $\Phi(\alpha_3)$ and thus our base case holds. Next, suppose that

$$\Phi^n(\beta_3) = \big([\alpha_3,\beta_3]\alpha_1\big)^n\beta_3\big(\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\big)^n$$

for some $n \in \mathbb{Z}_{>0}$. We wish to show that for $n+1$

$$\Phi^{n+1}(\beta_3) = \big([\alpha_3,\beta_3]\alpha_1\big)^{n+1}\beta_3\big(\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\big)^{n+1}$$

We begin by noting that

$$\Phi^{n+1}(\beta_3) = \Phi\big(\Phi^n(\beta_3)\big)$$

thus by our induction hypothesis we have

$$\begin{aligned}\Phi\big(\Phi^n(\beta_3)\big) &= \Phi\left(([\alpha_3,\beta_3]\alpha_1)^n\beta_3(\alpha_1^{-1}[\alpha_3,\beta_3]^{-1})^n\right)\\ &= \Phi\big(([\alpha_3,\beta_3]\alpha_1)^n\big)\Phi(\beta_3)\Phi\big((\alpha_1^{-1}[\alpha_3,\beta_3]^{-1})^n\big)\end{aligned}$$

Now by our previous lemmas this becomes

$$\begin{aligned}\Phi^{n+1}(\beta_3) &= \Phi\big(([\alpha_3,\beta_3]\alpha_1)^n\big)\Phi(\beta_3)\Phi\big((\alpha_1^{-1}[\alpha_3,\beta_3]^{-1})^n\big)\\ &= \big([\alpha_3,\beta_3]\alpha_1\big)^n[\alpha_3,\beta_3]\alpha_1\beta_3\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\big(\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\big)^n\\ &= \big([\alpha_3,\beta_3]\alpha_1\big)^{n+1}\beta_3\big(\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\big)^{n+1}\end{aligned}$$

and so we are done. □

### 4.3. **Computing** $\mathrm{Fix}\big((\Phi^n)^*\big)$.

4.3. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .

- $A_1$ :

$$\begin{aligned}A_1 &= \big([A_3,B_3]A_1\big)^n A_1\big(A_1^{-1}[A_3,B_3]^{-1}\big)^n\\ A_1 &= \big(A_1\big)^n A_1\big(A_1^{-1}\big)^n\\ A_1 &= A_1\big(A_1\big)^{n-1}A_1\big(A_1^{-1}\big)^n\\ A_1 &= A_1\end{aligned}$$

- $B_1$ :

$$\begin{aligned}B_1 &= B_1\big(A_1\big)^n\big(A_1^{-1}[A_3,B_3]^{-1}\big)^n\\ I &= \big(A_1\big)^n\big(A_1^{-1}[A_3,B_3]^{-1}\big)^n\\ \big([A_3,B_3]A_1\big)^n &= \big(A_1\big)^n\end{aligned}$$

- $A_3$ :

$$A_3 = \left(A_1[A_3, B_3]\right)^n A_3 \left([A_3, B_3]^{-1} A_1^{-1}\right)^n$$
$$A_3 = \left(A_1\right)^n A_3 \left(A_1^{-1}\right)^n$$
$$A_3 \left(A_1\right)^n = \left(A_1\right)^n A_3$$
$$\left[(A_1)^n, A_3\right] = I$$

- $B_3$ :

$$B_3 = \left(A_1[A_3, B_3]\right)^n B_3 \left([A_3, B_3]^{-1} A_1^{-1}\right)^n$$
$$B_3 = \left(A_1\right)^n B_3 \left(A_1^{-1}\right)^n$$
$$B_3 \left(A_1\right)^n = \left(A_1\right)^n B_3$$
$$\left[(A_1)^n, B_3\right] = I$$

Therefore, with the relations

$$\left([A_3, B_3]A_1\right)^n = \left(A_1\right)^n$$
$$\left[(A_1)^n, A_3\right] = I$$
$$\left[(A_1)^n, B_3\right] = I$$

our fixed point set for the map $\left(\Phi^n\right)^* = \left((T_{\gamma_1} \circ T_{\gamma_2}^{-1})^n\right)^*$ is defined as follows

$$\mathrm{Fix}\big((\Phi^n)^*\big) = \left\{(A_i, B_i) \in \mathrm{SU}(2)^6 \;:\; \prod_{i=1}^{3} \left[A_i, B_i\right] = I,\; \left([A_3, B_3]A_1\right)^n = \left(A_1\right)^n,\; \left[(A_1)^n, A_3\right] = I,\; \left[(A_1)^n, B_3\right] = I\right\}$$

Now, observe that $(A_1)^n$ is involved in three of our four relations, and that based on this $A_1$ essentially determines the other five entries. Thus define the map

$$\mu : \mathrm{Fix}\big((\Phi^n)^*\big) \longrightarrow \mathrm{SU}(2)$$

given by

$$\big(A_1, B_1, A_2, B_2, A_3, B_3\big) \longmapsto A_1$$

First, note that $\mu$ is a restriction of the projection map

$$\mathrm{p} : \mathrm{SU}(2)^6 \longrightarrow \mathrm{SU}(2)$$

which we know to be continuous as projections in the product topology are continuous, thus $\mu$ is continuous. Next we claim that $\mu$ is surjective.

**Lemma 9.** *The map*

$$\mu : \mathrm{Fix}\big((\Phi^n)^*\big) \longrightarrow \mathrm{SU}(2)$$
$$\big(A_1, B_1, A_2, B_2, A_3, B_3\big) \longmapsto A_1$$

*is surjective*

*Proof.* Fix $n \in \mathbb{Z} \setminus \{0\}$ and let $A_1 \in \mathrm{SU}(2)$. Now we wish to find corresponding $B_1, A_2, B_2, A_3, B_3 \in \mathrm{SU}(2)$ such that the 6-tuple lies in our fixed point set, that is

$$\big(A_1, B_1, A_2, B_2, A_3, B_3\big) \in \mathrm{Fix}\big((\Phi^n)^*\big)$$

To do so let us consider the relations that these matrices must satisfy

$$\textstyle\prod_{i=1}^{3} \left[A_i, B_i\right] = I$$
$$\left([A_3, B_3]A_1\right)^n = \left(A_1\right)^n$$
$$\left[(A_1)^n, A_3\right] = \left[(A_1)^n, B_3\right] = I$$

Observe that for the second and third relations, if we take both $A_3$ and $B_3$ to be the identity matrix then these two are satisfied. Furthermore, for the product of the commutator relation, with our choices of $A_3$ and $B_3$, this becomes

$$\big[A_1, B_1\big] = \big[A_2, B_2\big]^{-1}$$

Thus, we may choose our $B_1, A_2$ and $B_2$ such that this equality holds, for simplicity's sake again set them equal to the identity. With this 6-tuple

$$\big(A_1, I_{B_1}, I_{A_2}, I_{B_2}, I_{A_3}, I_{B_3}\big)$$

we have shown that it satisfies the relations in our fixed point set and thus for arbitrary $A_1 \in \mathrm{SU}(2)$ have found a corresponding element of our domain. Therefore, our map $\mu$ is surjective. □

Having shown that our map $\mu$ is a continuous surjection, we now wish to use it to classify our fixed point set. To do so we first define the following sets in SU(2) based on our possible values of $\big(A_1\big)^n$. Let

$$\mathcal{A}^n_{\not\pm} := \big\{A_1 \in \mathrm{SU}(2) \,:\, \big(A_1\big)^n \neq \pm I\big\}$$

be the set of all $A_1$ whose $n$-th power is non-central in SU(2) and let

$$\mathcal{A}^n_{\pm} := \big\{A_1 \in \mathrm{SU}(2) \,:\, \big(A_1\big)^n = \pm I\big\}$$

denote the set of all $A_1$ whose $n$-th power is central in SU(2). However, note that we can further partition this second set into the following two subsets,

$$\mathcal{A}^n_{+} := \big\{A_1 \in \mathrm{SU}(2) \,:\, \big(A_1\big)^n = I\big\}$$

and

$$\mathcal{A}^n_{-} := \big\{A_1 \in \mathrm{SU}(2) \,:\, \big(A_1\big)^n = -I\big\}$$

We may note that by construction, these three sets partition SU(2) as they represent the collection of fibers of the $n$-th power map. Now, with these sets we may consider their preimages under our map $\mu$, notably,

$$\mu^{-1}\big(\mathcal{A}^n_{\not\pm}\big) = \big\{(A_1, B_1, A_2, B_2, A_3, B_3) \in \mathrm{Fix}\big((\Phi^n)^*\big) \,:\, \big(A_1\big)^n \neq \pm I\big\}$$

and

$$\mu^{-1}\big(\mathcal{A}^n_{\pm}\big) = \big\{(A_1, B_1, A_2, B_2, A_3, B_3) \in \mathrm{Fix}\big((\Phi^n)^*\big) \,:\, \big(A_1\big)^n = \pm I\big\}$$

which we can represent as

$$\mu^{-1}\big(\mathcal{A}^n_{\pm}\big) = \mu^{-1}\big(\mathcal{A}^n_{+}\big) \cup \mu^{-1}\big(\mathcal{A}^n_{-}\big)$$

with

$$\mu^{-1}\big(\mathcal{A}^n_{+}\big) = \big\{(A_1, B_1, A_2, B_2, A_3, B_3) \in \mathrm{Fix}\big((\Phi^n)^*\big) \,:\, \big(A_1\big)^n = I\big\}$$

and

$$\mu^{-1}\big(\mathcal{A}^n_{-}\big) = \big\{(A_1, B_1, A_2, B_2, A_3, B_3) \in \mathrm{Fix}\big((\Phi^n)^*\big) \,:\, \big(A_1\big)^n = -I\big\}$$

Since $\mu$ is a continuous surjection and as previously noted our characterization sets, $\mathcal{A}^n_{\not\pm}$ and $\mathcal{A}^n_{\pm}$ partition SU(2), it follows that our fixed point set may be expressed as the union of these respective preimages, that is

$$\mathrm{Fix}\big((\Phi^n)^*\big) = \mu^{-1}\big(\mathcal{A}^n_{\not\pm}\big) \cup \mu^{-1}\big(\mathcal{A}^n_{\pm}\big) = \mu^{-1}\big(\mathcal{A}^n_{\not\pm}\big) \cup \big(\,\mu^{-1}\big(\mathcal{A}^n_{+}\big) \cup \mu^{-1}\big(\mathcal{A}^n_{-}\big)\big)$$

Therefore, in order to classify the connectedness of our fixed point set, it suffices to investigate the connectedness of these preimages. First, we will consider the preimage over the collection of $A_1$ such that their $n$-th power is a non-central element of SU(2). To do so lets fix $A_1 \in \mathcal{A}^n_{\not\pm}$. Then, we will consider our original relations from the fixed point set. We begin by noting that by our last two relations, we know that $\big(A_1\big)^n$ commutes with both $A_3$ and $B_3$. Thus, by definition $A_3$ and $B_3$ are in the centralizer of $\big(A_1\big)^n$. Since $\big(A_1\big)^n$ is assumed to be a non-central element, its centralizer is a maximal torus in SU(2). It follows that both $A_3$ and $B_3$ lie in this maximal torus. Recall that every maximal torus is abelian, hence, $A_3$ and $B_3$ must commute, that is

$$\big[A_3, B_3\big] = I$$

Now we may consider the product of the commutator relation, noting that with $A_3$ and $B_3$ commuting our expression simplifies as follows

$$\begin{aligned}\prod_{i=1}^{3}\big[A_i,B_i\big] &= I\\ \big[A_1,B_1\big]\big[A_2,B_2\big]\big[A_3,B_3\big] &= I\\ \big[A_1,B_1\big]\big[A_2,B_2\big] &= I\\ \big[A_1,B_1\big] &= \big[A_2,B_2\big]^{-1}\end{aligned}$$

Next lets examine our second relation. Observe that since the commutator of $A_3$ and $B_3$ is in the center of SU(2), then it commutes with $A_1$ and so we may distribute the exponent and realize that this relation is trivial. Therefore, for each $A_1 \in \mathcal{A}^n_{\not\pm}$ we can express its corresponding stratum of the fixed point set as

$$\{A_1\}\times\mathcal{F}^{\not\pm}_{A_1} := \{A_1\}\times\Big\{(B_1,A_2,B_2,A_3,B_3)\in \mathrm{SU}(2)^5 \,:\, \big[A_1,B_1\big]=\big[A_2,B_2\big]^{-1},\ \big[A_3,B_3\big]=\big[(A_1)^n,A_3\big]=\big[(A_1)^n,B_3\big]=I\Big\}$$

Hence, the preimage $\mu^{-1}\big(\mathcal{A}^n_{\not\pm}\big)$ can be represented by the union of over all such $A_1 \in \mathcal{A}^n_{\not\pm}$ of these corresponding fixed point stratum sets, that is

$$\mu^{-1}\big(\mathcal{A}^n_{\not\pm}\big) = \bigcup_{A_1\in\mathcal{A}^n_{\not\pm}} \{A_1\}\times\mathcal{F}^{\not\pm}_{A_1}$$

Next, we will consider the preimage over the collection $A_1$ such that their $n$-th power is a central element of SU(2). To do so lets fix $A_1 \in \mathcal{A}^n_{\pm}$. Then, we will consider our original relations from the fixed point set. We begin by considering the product of the commutators relation, noting that we cannot simplify this with the extra condition on $A_1$, and so we move on. Next lets examine our second relation. Observe that by our intial assumption on $A_1$ this relation simplifies to

$$\big([A_3,B_3]A_1\big)^n = \pm I$$

Finally, examining our last two relations, since $(A_1)^n$ commutes with every element of SU(2), then these relations are trivial. Thus for each $A_1 \in \mathcal{A}^n_{\pm}$ we can express its corresponding stratum of the fixed point set as

$$\{A_1\}\times\mathcal{F}^{\pm}_{A_1} := \{A_1\}\times\left\{(B_1,A_2,B_2,A_3,B_3)\in \mathrm{SU}(2)^5 \,:\, \prod_{i=1}^{3}\big[A_i,B_i\big]=I,\ \big([A_3,B_3]A_1\big)^n=\pm I\right\}$$

However, as we previously observed, we can split $\mathcal{A}^n_{\pm}$ into two subsets, $\mathcal{A}^n_{+}$ and $\mathcal{A}^n_{-}$. Therefore, for each $A_1' \in \mathcal{A}^n_{+}$ we can express its corresponding stratum of the fixed point set as

$$\{A_1'\}\times\mathcal{F}^{+}_{A_1'} := \{A_1'\}\times\left\{(B_1,A_2,B_2,A_3,B_3)\in \mathrm{SU}(2)^5 \,:\, \prod_{i=1}^{3}\big[A_i,B_i\big]=I,\ \big([A_3,B_3]A_1'\big)^n=I\right\}$$

Likewise, for each $A_1'' \in \mathcal{A}^n_{-}$ we can express its corresponding stratum of the fixed point set as

$$\{A_1''\}\times\mathcal{F}^{-}_{A_1''} := \{A_1''\}\times\left\{(B_1,A_2,B_2,A_3,B_3)\in \mathrm{SU}(2)^5 \,:\, \prod_{i=1}^{3}\big[A_i,B_i\big]=I,\ \big([A_3,B_3]A_1''\big)^n=-I\right\}$$

Now, note that these two fixed point stratum sets are disjoint as their corresponding characterization sets $\mathcal{A}^n_{+}$ and $\mathcal{A}^n_{-}$ are disjoint and so the $A_1$ arguments of each respective set cannot agree. Hence, the preimage $\mu^{-1}\big(\mathcal{A}^n_{\pm}\big)$ can be represented as the disjoint union of the two respective unions over all such $A_1' \in \mathcal{A}^n_{+}$ and $A_1'' \in \mathcal{A}^n_{-}$ of these corresponding fixed point stratum sets, that is

$$\mu^{-1}\big(\mathcal{A}^n_{\pm}\big) = \bigcup_{A_1'\in\mathcal{A}^n_{+}} \{A_1'\}\times\mathcal{F}^{+}_{A_1'} \ \sqcup \bigcup_{A_1''\in\mathcal{A}^n_{-}} \{A_1''\}\times\mathcal{F}^{-}_{A_1''}$$

Now that we have described each respective preimage of $\mu$ which partition our fixed point set. We may observe that our fixed point set can be represented as the disjoint union of three respective unions of our fixed point stratum sets, $\{A_1\}\times\mathcal{F}^{\not\equiv}_{A_1}$, $\{A_1'\}\times\mathcal{F}^{+}_{A_1'}$, and $\{A_1''\}\times\mathcal{F}^{-}_{A_1''}$, over our $A_1$ characterization sets, $\mathcal{A}^n_{\not\equiv}$, $\mathcal{A}^n_{+}$, and $\mathcal{A}^n_{-}$. That is, from our original representation

$$\mathrm{Fix}\big((\Phi^n)^*\big) = \mu^{-1}\big(\mathcal{A}^n_{\not\equiv}\big)\ \cup\ \big(\mu^{-1}\big(\mathcal{A}^n_{+}\big)\ \cup\ \mu^{-1}\big(\mathcal{A}^n_{-}\big)\big)$$

we have that

$$\mathrm{Fix}\big((\Phi^n)^*\big) = \bigcup_{A_1\in\mathcal{A}^n_{\not\equiv}}\{A_1\}\times\mathcal{F}^{\not\equiv}_{A_1}\ \cup\ \bigcup_{A_1'\in\mathcal{A}^n_{+}}\{A_1'\}\times\mathcal{F}^{+}_{A_1'}\ \cup\ \bigcup_{A_1''\in\mathcal{A}^n_{-}}\{A_1''\}\times\mathcal{F}^{-}_{A_1''}$$

Due note that this is technically a disjoint union of these respective unions as by construction the $A_1$ argument of each respective collection of fixed point stratum sets cannot agree. Therefore, as we attempt to classify the connectedness of our fixed point set for arbitrary powers of $n$, it suffices to determine the connectedness of each collection of fixed point stratum sets respectively. We will begin by investigating the connectedness of the preimage of our characterization set $\mathcal{A}^n_{\not\equiv}$, that is

$$\mu^{-1}\big(\mathcal{A}^n_{\not\equiv}\big) = \bigcup_{A_1\in\mathcal{A}^n_{\not\equiv}}\{A_1\}\times\mathcal{F}^{\not\equiv}_{A_1}$$

To do so we first consider each individual fixed point stratum set $\{A_1\}\times\mathcal{F}^{\not\equiv}_{A_1}$.

**Lemma 10.** *$\{A_1\}\times\mathcal{F}^{\not\equiv}_{A_1}$ is connected for every $A_1\in\mathcal{A}^n_{\not\equiv}$, $n\in\mathbb{Z}\setminus\{0\}$*

*Proof.* Fix $n\in\mathbb{Z}\setminus\{0\}$ and fix $A_1\in\mathcal{A}^n_{\not\equiv}$. With

$$\{A_1\}\times\mathcal{F}^{\not\equiv}_{A_1} := \{A_1\}\times\Big\{(B_1,A_2,B_2,A_3,B_3)\in\mathrm{SU}(2)^5\ :\ \big[A_1,B_1\big]=\big[A_2,B_2\big]^{-1},\ \big[A_3,B_3\big]=\big[(A_1)^n,A_3\big]=\big[(A_1)^n,B_3\big]=I\Big\}$$

we begin by noting that the singleton set $\{A_1\}$ is connected in $\mathrm{SU}(2)$ and the product

$$\{A_1\}\times\mathcal{F}^{\not\equiv}_{A_1}$$

is homeomorphic to $\mathcal{F}^{\not\equiv}_{A_1}$, as it is just a copy of this set at $A_1$. Thus to determine the connectedness of the fixed point stratum set, it suffices to show the connectedness of $\mathcal{F}^{\not\equiv}_{A_1}$. With this, we first define

$$\mathcal{B}^{\not\equiv}_{A_1} := \big\{(B_1,A_3,B_3)\in\mathrm{SU}(2)^3\ :\ \big[A_3,B_3\big]=\big[(A_1)^n,A_3\big]=\big[(A_1)^n,B_3\big]=I\big\}$$

Now consider the map

$$\pi:\mathcal{F}^{\not\equiv}_{A_1}\ \longrightarrow\ \mathcal{B}^{\not\equiv}_{A_1}$$

given by

$$\big(B_1,A_2,B_2,A_3,B_3\big)\ \longmapsto\ \big(B_1,A_3,B_3\big)$$

Note that $\pi$ is a restriction of the projection map

$$\mathrm{p}:\mathrm{SU}(2)^5\ \longrightarrow\ \mathrm{SU}(2)^3$$

which we know to be continuous as projections in the product topology are continuous, thus $\pi$ is continuous. Additionally, we claim that $\pi$ is surjective. To see this, take $\big(B_1,A_3,B_3\big)\in\mathcal{B}^{\not\equiv}_{A_1}$. By the definition of $\mathcal{B}^{\not\equiv}_{A_1}$, we have

$$\big[A_3,B_3\big]=I\qquad\big[(A_1)^n,A_3\big]=I\qquad\big[(A_1)^n,B_3\big]=I$$

Thus our goal is to find an $A_2,B_2\in\mathrm{SU}(2)$ such that the 5-tuple $\big(B_1,A_2,B_2,A_3,B_3\big)$ lies in $\mathcal{F}^{\not\equiv}_{A_1}$. That is

$$\big[A_1,B_1\big]=\big[A_2,B_2\big]^{-1}$$

Define

$$A_2 := A_1^{-1}\quad\text{and}\quad B_2 := B_1^{-1},$$

note that since $\mathrm{SU}(2)$ is a group, then $A_2,B_2\in\mathrm{SU}(2)$. Now, observe that

$$\big[A_1,B_1\big]=A_1B_1A_1^{-1}B_1^{-1}=A_2^{-1}B_2^{-1}A_2B_2=\big[A_2,B_2\big]^{-1}$$

Hence along with the assumed commutation relations in $\mathcal{B}^{\neq}_{A_1}$, it follows that

$$\big(B_1, A_2, B_2, A_3, B_3\big) \in \mathcal{F}^{\neq}_{A_1}$$

and thus

$$\pi\big(B_1, A_2, B_2, A_3, B_3\big) = \big(B_1, A_3, B_3\big)$$

so $\pi$ is surjective. Now that we have established that $\pi$ is a continuous surjection, let us consider the fibers of this map. Given a point $\big(B_1, A_3, B_3\big) \in \mathcal{B}^{\neq}_{A_1}$, the fiber over this point is defined as

$$\pi^{-1}\big(B_1, A_3, B_3\big) = \big\{(B_1, A_2, B_2, A_3, B_3) \in \mathcal{F}^{\neq}_{A_1}\big\}$$

Note that since in $\mathcal{F}^{\neq}_{A_1}$ the only relation involving $\big(A_2, B_2\big)$ is

$$\big[A_2, B_2\big] = \big[A_1, B_1\big]^{-1}$$

we may fix $\big(B_1, A_3, B_3\big) \in \mathcal{B}^{\neq}_{A_1}$ and define the map

$$\psi : \pi^{-1}\big(B_1, A_3, B_3\big) \longrightarrow \big\{(A_2, B_2) \in \mathrm{SU}(2)^2 \,:\, [A_2, B_2] = [A_1, B_1]^{-1}\big\}$$

given by

$$\big(B_1, A_2, B_2, A_3, B_3\big) \longmapsto \big(A_2, B_2\big)$$

Since we fixed our fiber, acting as the domain, by the uniqueness of inverses in SU(2) our map $\psi$ is well-defined and surjective. Moreover, by this uniqueness property and the fact that we fixed our $B_1, A_3$ and $B_3$ arguments, we must have that the map is injective, thus $\psi$ is a bijection. Now we observe that $\psi$ is a restriction of the projection map

$$\mathrm{p} : \mathrm{SU}(2)^5 \longrightarrow \mathrm{SU}(2)^2$$

which we know to be continuous as projections in the product topology are continuous, thus $\psi$ is continuous. Furthermore, $\psi^{-1}$ is a restriction of the inclusion map

$$\imath : \mathrm{SU}(2)^2 \longrightarrow \mathrm{SU}(2)^5$$

which we know to be continuous as inclusions in the product topology are continuous, thus $\psi^{-1}$ is continuous. Therefore, $\psi$ is a homeomorphism, that is

$$\pi^{-1}\big(B_1, A_3, B_3\big) \cong \big\{(A_2, B_2) \in \mathrm{SU}(2)^2 \,:\, [A_2, B_2] = [A_1, B_1]^{-1}\big\}.$$

Importantly, observe that

$$\big\{(A_2, B_2) \in \mathrm{SU}(2)^2 \,:\, [A_2, B_2] = [A_1, B_1]^{-1}\big\}$$

is just a fiber over the commutator map of SU(2) and thus is connected. Since we have shown that an arbitrary fiber, $\pi^{-1}(B_1, A_3, B_3)$, is connected, it follows that the fibers of our map $\pi$ are connected. Therefore, $\pi$ is a continuous, surjective map with connected fibers from our total space $\mathcal{F}^{\neq}_{A_1}$ into our defined base space $\mathcal{B}^{\neq}_{A_1}$. Thus, we will now investigate the connectedness of this base space $\mathcal{B}^{\pm}_{A_1}$. To do so let us first examine the commutation relations in $\mathcal{B}^{\neq}_{A_1}$, noting that

$$\big[A_3, B_3\big] = \big[(A_1)^n, A_3\big] = \big[(A_1)^n, B_3\big] = I$$

implies that $A_3, B_3$ commute. Recall that any two elements of SU(2) commute if and only if they lie in the same maximal torus, which in SU(2) is conjugate to the subgroup of diagonal matrices. Let T denote a maximal torus in SU(2),

$$\mathrm{T} = \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} : \theta \in [0, 2\pi) \right\}$$

Since $A_3, B_3$ commute, there exists $g \in \mathrm{SU}(2)$ such that

$$g\, A_3 g^{-1},\ g\, B_3 g^{-1} \in \mathrm{T}$$

This follows from the fact that all maximal tori in SU(2) are conjugate and every element is contained in some maximal torus. Therefore, the set of commuting 2-tuples in SU(2) is

$$\begin{aligned}\mathcal{C}^{\not\equiv}_{A_1} &:= \{(a_3, b_3) \in \mathrm{SU}(2)^2 \,:\, [a_3, b_3] = I\} \\ &= \{(g\,t_1 g^{-1}, g\,t_2 g^{-1}) \,:\, g \in \mathrm{SU}(2),\ t_1, t_2 \in \mathrm{T}\}\end{aligned}$$

We wish to show that $\mathcal{C}^{\not\equiv}_{A_1}$ is connected, in order to do so we define the map

$$\Omega : \mathrm{SU}(2) \times \mathrm{T}^2 \longrightarrow \mathcal{C}^{\not\equiv}_{A_1}$$

by

$$\Omega\big(g, (t_1, t_2)\big) = \big(g\,t_1 g^{-1},\ g\,t_2 g^{-1}\big).$$

Observe that the domain of the map, $\mathrm{SU}(2) \times \mathrm{T}^2$, is connected since SU(2) is connected, T is connected, and the finite product of connected spaces is connected. Therefore, it suffices to show that $\Omega$ is continuous and surjective, as the image of a connected space under a continuous map is connected. We begin by verifying the surjectivity of $\Omega$. Let $(a_3, b_3) \in \mathcal{C}^{\not\equiv}_{A_1}$, then there exists $g \in \mathrm{SU}(2)$ such that

$$g^{-1} a_3 g,\ g^{-1} b_3 g \in \mathrm{T}.$$

This follows from the fact that all maximal tori in SU(2) are conjugate and every element is contained in some maximal torus. Now, if we set

$$t_1 = g^{-1} a_3 g \qquad \text{and} \qquad t_2 = g^{-1} b_3 g.$$

then every element of $\mathcal{C}^{\not\equiv}_{A_1}$ is in the image of $\mathrm{SU}(2) \times \mathrm{T}^2$ under $\Omega$ and so the map is surjective. For the continuity of the map, we note that $\Omega$ is defined on group operations, multiplication and inversion, which are smooth in SU(2), and so the map is continuous. Hence, $\mathcal{C}^{\not\equiv}_{A_1}$ is connected. Now, there are no relations involving $B_1$ in the definition of $\mathcal{B}^{\not\equiv}_{A_1}$, therefore, for any fixed commuting 2-tuple $(A_3, B_3)$, $B_1$ can be any element in SU(2). Thus,

$$\mathcal{B}^{\not\equiv}_{A_1} \cong \mathcal{C}^{\not\equiv}_{A_1} \times \mathrm{SU}(2).$$

We know that SU(2) is connected and we have just shown that $\mathcal{C}^{\not\equiv}_{A_1}$ is connected, therefore, since the finite product of connected spaces is connected, it follows that $\mathcal{B}^{\not\equiv}_{A_1}$ is connected. To recap, we have shown that there is a continuous surjection

$$\pi : \mathcal{F}^{\not\equiv}_{A_1} \longrightarrow \mathcal{B}^{\not\equiv}_{A_1}$$

with non-empty, connected fibers. Therefore, since the base space $\mathcal{B}^{\not\equiv}_{A_1}$ is connected, it follows that $\mathcal{F}^{\not\equiv}_{A_1}$ is connected. Hence the entire fixed point stratum set

$$\{A_1\} \times \mathcal{F}^{\not\equiv}_{A_1}$$

is connected. □

We have now shown that for our preimage over the characterization set $\mathcal{A}^n_{\not\equiv}$ each fixed point stratum set is connected. So to determine the connectedness of $\mu^{-1}\big(\mathcal{A}^n_{\not\equiv}\big)$ we need to consider the union of all such fixed point stratum sets over our characterization set.

**Lemma 11.** *$\mu^{-1}\big(\mathcal{A}^n_{\not\equiv}\big) = \bigcup_{A_1 \in \mathcal{A}^n_{\not\equiv}} \{A_1\} \times \mathcal{F}^{\not\equiv}_{A_1}$ has $|n|$ connected components for every $n \in \mathbb{Z} \setminus \{0\}$*

*Proof.* Fix $n \in \mathbb{Z} \setminus \{0\}$. We begin by noting that by the previous lemma, we know that for each $A_1 \in \mathcal{A}^n_{\not\pm}$ our corresponding fixed point stratum set

$$\{A_1\} \times \mathcal{F}^{\not\pm}_{A_1} := \{A_1\} \times \Big\{(B_1, A_2, B_2, A_3, B_3) \in \mathrm{SU}(2)^5 \,:\, [A_1, B_1] = [A_2, B_2]^{-1},\ [A_3, B_3] = [(A_1)^n, A_3] = [(A_1)^n, B_3] = I\Big\}$$

is connected. Thus the remaining determination of the connectedness of our preimage

$$\mu^{-1}\big(\mathcal{A}^n_{\not\pm}\big) = \bigcup_{A_1 \in \mathcal{A}^n_{\not\pm}} \{A_1\} \times \mathcal{F}^{\not\pm}_{A_1}$$

depends on that of the parameter space, our characterization set $\mathcal{A}^n_{\not\pm}$. Now with

$$\mathcal{A}^n_{\not\pm} = \big\{A_1 \in \mathrm{SU}(2) \,:\, \big(A_1\big)^n \neq \pm I\big\}$$

let us recall that this is merely the union of fibers of the $n$-th power map of SU(2). Thus we will consider this map,

$$\mathrm{p}_n : \mathrm{SU}(2) \longrightarrow \mathrm{SU}(2)$$

given by

$$Z \longmapsto Z^n$$

Observe that since $\mathrm{p}_n$ is surjective

$$\mathcal{A}^n_{\not\pm} = \mathrm{SU}(2) \setminus \big(\mathrm{p}_n^{-1}(I) \cup \mathrm{p}_n^{-1}(-I)\big)$$

Thus let us examine

$$\mathrm{SU}(2) \setminus \big(\mathrm{p}_n^{-1}(I) \cup \mathrm{p}_n^{-1}(-I)\big)$$

Note that for any $W \in \mathrm{SU}(2) \setminus \big(\mathrm{p}_n^{-1}(I) \cup \mathrm{p}_n^{-1}(-I)\big)$, $W$ is diagonalizable and can be written up to conjugation as

$$W \sim \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}, \quad \theta \in [0, \pi]$$

Now this represents the conjugacy class determined by the eigenvalues $(e^{i\theta}, e^{-i\theta})$, which by our condition on $W$ that

$$\big(W\big)^n \neq I$$

must satisfy

$$\big(e^{i\theta}\big)^n \neq \pm 1 \implies e^{in\theta} \neq \pm 1 \iff n\theta \not\equiv 0 \pmod{\pi}$$

Therefore, we have

$$\theta \neq \frac{\pi k}{n}, \quad k \in \mathbb{Z}$$

Due to the equivalence under conjugation of $\theta \sim -\theta$ and $\theta \sim \theta + 2\pi$ in SU(2), we only consider $\theta \in [0, \pi]$. Removing all such angles from our interval $[0, \pi]$ for $0 \leq k \leq n$ we are left with $n$ open intervals

$$\left(0, \frac{\pi}{n}\right), \left(\frac{\pi}{n}, \frac{2\pi}{n}\right), \ldots, \left(\frac{(n-1)\pi}{n}, \pi\right).$$

Now, we claim that each of these open intervals corresponds to a connected component of

$$\mathrm{SU}(2) \setminus \big(\mathrm{p}_n^{-1}(I) \cup \mathrm{p}_n^{-1}(-I)\big)$$

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[INSERT] DR. DUNCAN: Implicit/Inverse Function Theorem Argument

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□

With this we have shown that $\mu^{-1}\big(\mathcal{A}^n_{\not\pm}\big)$ has $|n|$ connected components for every $n \in \mathbb{Z} \setminus \{0\}$. Thus it is left to classify the connectedness of our two remaining preimages which partition our fixed point set. To do so, first we will consider our fixed point stratum sets which arise in the case where the $n$-th power of $A_1$ is in the center of SU(2), specifically when it is equal to the identity.

**Lemma 12.** *For every $n \in \mathbb{Z} \setminus \{0\}$ and $A_1 \in \mathcal{A}_+^n$, $\{A_1\} \times \mathcal{F}_{A_1}^+$ has*

$$\left\lfloor \frac{|n|}{2} \right\rfloor + 1$$

*connected components*

*Proof.* Fix $n \in \mathbb{Z} \setminus \{0\}$ and fix $A_1 \in \mathcal{A}_+^n$. With

$$\{A_1\} \times \mathcal{F}_{A_1}^+ := \big\{A_1\big\} \times \left\{ (B_1, A_2, B_2, A_3, B_3) \in \mathrm{SU}(2)^5 \, : \, \prod_{i=1}^{3} \big[A_i, B_i\big] = I, \; \big([A_3, B_3]A_1\big)^n = I \right\}$$

we begin by noting that the singleton set $\{A_1\}$ is connected in SU(2) and the product

$$\{A_1\} \times \mathcal{F}_{A_1}^+$$

is homeomorphic to $\mathcal{F}_{A_1}^+$ as it is just a copy of this set at $A_1$. Thus to determine the connectedness of the fixed point stratum set, it suffices to show the connectedness of $\mathcal{F}_{A_1}^+$. With this, we first define

$$\mathcal{B}_{A_1}^+ := \big\{(B_1, A_3, B_3) \in \mathrm{SU}(2)^4 \, : \, ([A_3, B_3]A_1)^n = I\big\}$$

Now consider the map

$$\pi : \mathcal{F}_{A_1}^+ \; \longrightarrow \; \mathcal{B}_{A_1}^+$$

given by

$$\big(B_1, A_2, B_2, A_3, B_3\big) \; \longmapsto \; \big(B_1, A_3, B_3\big)$$

Note that $\pi$ is a restriction of the projection map

$$\mathrm{p} : \mathrm{SU}(2)^5 \; \longrightarrow \; \mathrm{SU}(2)^3$$

which we know to be continuous as projections in the product topology are continuous, thus $\pi$ is continuous. Additionally, we claim that $\pi$ is surjective. To see this, take $\big(B_1, A_3, B_3\big) \in \mathcal{B}_{A_1}^+$. Then by the definition of $\mathcal{B}_{A_1}^+$, we have

$$\big([A_3, B_3]A_1\big)^n = I$$

Thus our goal is to find an $A_2, B_2 \in \mathrm{SU}(2)$ such that the 5-tuple $\big(B_1, A_2, B_2, A_3, B_3\big)$ lies in $\mathcal{F}_{A_1}^+$, that is

$$\big[A_2, B_2\big] = \big[A_1, B_1\big]^{-1}\big[A_3, B_3\big]^{-1}$$

Recall that every element in SU(2) can be expressed as a commutator. Hence, because

$$\big[A_1, B_1\big]^{-1}\big[A_3, B_3\big]^{-1} \in \mathrm{SU}(2)$$

then by the surjectivity of the commutator map in SU(2), there exists an $X$, $Y \in \mathrm{SU}(2)$ such that

$$\big[X, Y\big] = \big[A_1, B_1\big]^{-1}\big[A_3, B_3\big]^{-1}$$

Define

$$A_2 := X \quad \text{and} \quad B_2 := Y.$$

Now, observe that

$$\begin{aligned} &\big[A_1, B_1\big]\big[X, Y\big]\big[A_3, B_3\big] \\ &= \big[A_1, B_1\big]\big(\big[A_1, B_1\big]^{-1}\big[A_3, B_3\big]^{-1}\big)\big[A_3, B_3\big] \\ &= \big(\big[A_1, B_1\big]\big[A_1, B_1\big]^{-1}\big)\big(\big[A_3, B_3\big]^{-1}\big[A_3, B_3\big]\big) \\ &= I \end{aligned}$$

Hence along with the relation in $\mathcal{B}_{A_1}^+$, it follows that

$$\big(B_1, A_2, B_2, A_3, B_3\big) \in \mathcal{F}_{A_1}^+$$

and thus

$$\pi\big(B_1, A_2, B_2, A_3, B_3\big) = \big(B_1, A_3, B_3\big)$$

so $\pi$ is surjective. Now that we have established that $\pi$ is a continuous surjection, let us consider the fibers of this map. Given a point $\big(B_1, A_3, B_3\big) \in \mathcal{B}^+_{A_1}$, the fiber over this point is defined as

$$\pi^{-1}\big(B_1, A_3, B_3\big) = \big\{(B_1, A_2, B_2, A_3, B_3) \in \mathcal{F}^+_{A_1}\big\}$$

Note that since in $\mathcal{F}^+_{A_1}$ the only relation involving $\big(A_2, B_2\big)$ is the product of the commutators, which we can express as

$$\big[A_2, B_2\big] = \big[A_1, B_1\big]^{-1}\big[A_3, B_3\big]^{-1}$$

we may fix $\big(B_1, A_3, B_3\big) \in \mathcal{B}^+_{A_1}$ and define the map

$$\psi : \pi^{-1}\big(B_1, A_3, B_3\big) \;\longrightarrow\; \big\{(A_2, B_2) \in \mathrm{SU}(2)^2 \,:\, [A_2, B_2] = [A_1, B_1]^{-1}[A_3, B_3]^{-1}\big\}$$

given by

$$\big(B_1, A_2, B_2, A_3, B_3\big) \;\longmapsto\; \big(A_2, B_2\big)$$

Since we fixed our fiber, acting as the domain, by the uniqueness of inverses in SU(2) our map $\psi$ is well-defined and surjective. Moreover, by this uniqueness property and the fact that we fixed our $B_1, A_3$ and $B_3$ arguments, we must have that the map is injective, thus $\psi$ is a bijection. Now we observe that $\psi$ is a restriction of the projection map

$$\mathrm{p} : \mathrm{SU}(2)^5 \;\longrightarrow\; \mathrm{SU}(2)^2$$

which we know to be continuous as projections in the product topology are continuous, thus $\psi$ is continuous. Furthermore, $\psi^{-1}$ is a restriction of the inclusion map

$$\mathfrak{i} : \mathrm{SU}(2)^2 \;\longrightarrow\; \mathrm{SU}(2)^5$$

which we know to be continuous as inclusions in the product topology are continuous, thus $\psi^{-1}$ is continuous. Therefore, $\psi$ is a homeomorphism, that is

$$\pi^{-1}\big(B_1, A_3, B_3\big) \cong \big\{(A_2, B_2) \in \mathrm{SU}(2)^2 \,:\, [A_2, B_2] = [A_1, B_1]^{-1}[A_3, B_3]^{-1}\big\}$$

Importantly, observe that

$$\big\{(A_2, B_2) \in \mathrm{SU}(2)^2 \,:\, [A_2, B_2] = [A_1, B_1]^{-1}[A_3, B_3]^{-1}\big\}$$

is just a fiber over the commutator map of SU(2) and thus is connected. Since we have shown that an arbitrary fiber, $\pi^{-1}(B_1, A_3, B_3)$, is connected, it follows that the fibers of our map $\pi$ are connected. Therefore, $\pi$ is a continuous, surjective map with connected fibers from our total space $\mathcal{F}^+_{A_1}$ into our defined base space $\mathcal{B}^+_{A_1}$. Thus, we will now investigate the connectedness of this base space $\mathcal{B}^+_{A_1}$. To do so consider the $n$-th power map of SU(2)

$$\mathrm{p}_n : \mathrm{SU}(2) \;\longrightarrow\; \mathrm{SU}(2)$$

given by

$$Z \;\longmapsto\; Z^n$$

Let us examine the fiber over the identity

$$\mathrm{p}_n^{-1}(I) = \big\{X \in \mathrm{SU}(2) \,:\, (X)^n = I\big\}$$

Note that for any $X \in \mathrm{p}_n^{-1}(I)$, $X$ is a matrix in SU(2), so it is diagonalizable and can be written up to conjugation as

$$X \sim \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}, \quad \theta \in [0, \pi]$$

Now this represents the conjugacy class determined by the eigenvalues $(e^{i\theta}, e^{-i\theta})$, which by our condition on $X$ that

$$\big(X\big)^n = I$$

must satisfy

$$\left(e^{i\theta}\right)^n = 1 \implies e^{in\theta} = 1 \iff n\theta \equiv 0 \pmod{2\pi}$$

Therefore, we have

$$\theta = \frac{2\pi k}{n}, \quad k \in \mathbb{Z}$$

However, due to the equivalence under conjugation of $\theta \sim -\theta$ and $\theta \sim \theta + 2\pi$ in SU(2), we only consider $\theta \in [0, \pi]$. Thus,

$$0 \leq \frac{2\pi k}{|n|} \leq \pi \implies 0 \leq k \leq \left\lfloor \frac{|n|}{2} \right\rfloor$$

and so $k = 0, 1, \ldots, \left\lfloor \frac{|n|}{2} \right\rfloor$. Hence our possible eigenvalue pairs are

$$\left\{ e^{\frac{i2\pi k}{|n|}}, e^{\frac{-i2\pi k}{|n|}} \right\} \quad \text{where} \quad k = 0, 1, \ldots, \left\lfloor \frac{|n|}{2} \right\rfloor$$

Since in SU(2), two matrices are conjugate if and only if they have the same eigenvalues, these pairs form distinct conjugacy classes. So this fiber is equal to the union of the conjugacy classes of the diagonal matrices with these distinct eigenvalues, that is

$$\bigcup_{k=0}^{\left\lfloor \frac{|n|}{2} \right\rfloor} \mathcal{D}_k^+$$

where

$$\mathcal{D}_k^+ := \left\{ U \text{diag}\left( e^{\frac{i2\pi k}{|n|}}, e^{\frac{-i2\pi k}{|n|}} \right) U^{-1} \, : \, U \in \text{SU}(2) \right\}$$

Now, from our relation we know that $[A_3, B_3]A_1$ lies in this fiber of the $n$-th power map, thus

$$[A_3, B_3]A_1 \in \bigcup_{k=0}^{\left\lfloor \frac{|n|}{2} \right\rfloor} \mathcal{D}_k^+$$

Recall that the single element class $\{I\}$ is connected and that every conjugacy class of a non-central element in SU(2) is connected and homeomorphic to $S^2$. Therefore we know that

$$\mathcal{D}_k^+ \quad \text{is connected for every} \quad k = 0, 1, \ldots, \left\lfloor \frac{|n|}{2} \right\rfloor$$

However, it is important to note that

$$\bigcup_{k=0}^{\left\lfloor \frac{|n|}{2} \right\rfloor} \mathcal{D}_k^+$$

is a disjoint union, thus consists of $\left\lfloor \frac{|n|}{2} \right\rfloor + 1$ connected components. Therefore

$$\bigcup_{k=0}^{\left\lfloor \frac{|n|}{2} \right\rfloor} \mathcal{D}_k^+ \cong \{I\} \sqcup \underbrace{S^2 \sqcup \cdots \sqcup S^2}_{\left\lfloor \frac{|n|}{2} \right\rfloor \text{ times}}$$

Now note that $B_1$ is not involved in the relation in $\mathcal{B}_{A_1}^+$ and so it is unconstrained in SU(2). We do have, however, that $A_3$ and $B_3$ are involved this relation, thus we will consider the map

$$\Lambda : \text{SU}(2)^2 \longrightarrow \text{SU}(2)$$

given by

$$\left(A_3, B_3\right) \longmapsto \left[A_3, B_3\right] A_1$$

Observe that this is a continuous map as it is defined on a group operation, multiplication, which is smooth in SU(2). The key observation is that by our relation on $\mathcal{B}^+_{A_1}$ we have that

$$\big(B_1, A_3, B_3\big) \in \mathcal{B}^+_{A_1} \iff \Lambda\big(A_3, B_3\big) \in \mathrm{p}_n^{-1}(I)$$

Therefore,

$$\mathcal{B}^+_{A_1} = \mathrm{SU}(2) \times \Lambda^{-1}\big(\mathrm{p}_n^{-1}(I)\big)$$

and hence we can express $\mathcal{B}^+_{A_1}$ as the disjoint union of $\left\lfloor \frac{|n|}{2} \right\rfloor + 1$ subsets, that is

$$\mathcal{B}^+_{A_1} = \mathrm{SU}(2) \times \left( \bigcup_{k=0}^{\left\lfloor \frac{|n|}{2} \right\rfloor} \left[\Lambda^{-1}\big(\mathcal{D}^+_k\big)\right] \right)$$

Since $\Lambda$ is continuous and the continuous preimage of a connected set is connected, it follows that

$$\Lambda^{-1}\big(\mathcal{D}^+_k\big)$$

is connected for each $k = 0, 1, \ldots, \left\lfloor \frac{|n|}{2} \right\rfloor$. Moreover, since SU(2) is connected and the finite product of a connected spaces is connected, it follows that for each $k = 0, 1, \ldots, \left\lfloor \frac{|n|}{2} \right\rfloor$,

$$\mathrm{SU}(2) \times \Lambda^{-1}\big(\mathcal{D}^+_k\big)$$

is connected. Therefore, our set $\mathcal{B}^+_{A_1}$ has $\left\lfloor \frac{|n|}{2} \right\rfloor + 1$ connected components. So we have shown that there is a continuous surjection

$$\pi : \mathcal{F}^+_{A_1} \longrightarrow \mathcal{B}^+_{A_1}$$

with non-empty, connected fibers. Since every fiber is non-empty and connected, and the base space $\mathcal{B}^+_{A_1}$ has $\left\lfloor \frac{|n|}{2} \right\rfloor + 1$ connected components, it follows that $\mathcal{F}^+_{A_1}$ has $\left\lfloor \frac{|n|}{2} \right\rfloor + 1$ connected components. Hence the entire fixed point stratum set

$$\{A_1\} \times \mathcal{F}^+_{A_1}$$

has $\left\lfloor \frac{|n|}{2} \right\rfloor + 1$ connected components. □

Having classified the connectedness of each individual fixed point stratum set corresponding to the fibers of $A_1 \in \mathcal{A}^n_+$ under $\mu$, we now wish to consider the union so that we know the number of connected components of the entire preimage.

**Lemma 13.** *For every $n \in \mathbb{Z} \setminus \{0\}$,*

$$\mu^{-1}\big(\mathcal{A}^n_+\big) = \bigcup_{A_1 \in \mathcal{A}^n_+} \{A_1\} \times \mathcal{F}^+_{A_1}$$

*has*

$$\left(\left\lfloor \frac{|n|}{2} \right\rfloor + 1\right)^2$$

*connected components.*

*Proof.* Fix $n \in \mathbb{Z} \setminus \{0\}$. We begin by noting that by the previous lemma, we know that for each $A_1 \in \mathcal{A}^n_+$ our corresponding fixed point stratum set

$$\{A_1\} \times \mathcal{F}^+_{A_1} := \big\{A_1\big\} \times \left\{ (B_1, A_2, B_2, A_3, B_3) \in \mathrm{SU}(2)^5 \ : \ \prod_{i=1}^{3} \big[A_i, B_i\big] = I, \ \big([A_3, B_3]A_1\big)^n = I \right\}$$

has $\left\lfloor \frac{|n|}{2} \right\rfloor + 1$ connected components. Thus the remaining determination of the connectedness of our preiamge

$$\mu^{-1}(\mathcal{A}_+^n) = \bigcup_{A_1 \in \mathcal{A}_+^n} \{A_1\} \times \mathcal{F}_{A_1}^+$$

depends on that of the parameter space, our characterization set $\mathcal{A}_+^n$. Now with

$$\mathcal{A}_+^n = \left\{ A_1 \in \mathrm{SU}(2) \,:\, \left(A_1\right)^n = I \right\}$$

let us recall that this is merely the fiber of the identity under the $n$-th power map of SU(2). Thus we will consider this fiber, first defining the $n$-th power map,

$$\mathrm{p}_n : \mathrm{SU}(2) \longrightarrow \mathrm{SU}(2)$$

given by

$$Z \longmapsto Z^n$$

Now the fiber of the identity under this map is simply

$$\mathrm{p}_n^{-1}(I) = \left\{ X \in \mathrm{SU}(2) \,:\, (X)^n = I \right\}$$

Now, for any $X \in \mathrm{p}_n^{-1}(I)$, $X$ is a matrix in SU(2), so it is diagonalizable and can be written up to conjugation as

$$X \sim \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}, \quad \theta \in [0, \pi]$$

Now this represents the conjugacy class determined by the eigenvalues $(e^{i\theta}, e^{-i\theta})$, which by our condition on $X$ that

$$\left(X\right)^n = I$$

must satisfy

$$\left(e^{i\theta}\right)^n = 1 \implies e^{in\theta} = 1 \iff n\theta \equiv 0 \pmod{2\pi}$$

Therefore, we have

$$\theta = \frac{2\pi k}{n}, \quad k \in \mathbb{Z}$$

However, due to the equivalence under conjugation of $\theta \sim -\theta$ and $\theta \sim \theta + 2\pi$ in SU(2), we only consider $\theta \in [0, \pi]$. Thus,

$$0 \leq \frac{2\pi k}{|n|} \leq \pi \implies 0 \leq k \leq \left\lfloor \frac{|n|}{2} \right\rfloor$$

and so $k = 0, 1, \ldots, \left\lfloor \frac{|n|}{2} \right\rfloor$. Hence our possible eigenvalue pairs are

$$\left\{ e^{\frac{i2\pi k}{|n|}}, e^{\frac{-i2\pi k}{|n|}} \right\} \quad \text{where} \quad k = 0, 1, \ldots, \left\lfloor \frac{|n|}{2} \right\rfloor$$

Since in SU(2), two matrices are conjugate if and only if they have the same eigenvalues, these pairs form distinct conjugacy classes. So this fiber is equal to the union of the conjugacy classes of the diagonal matrices with these distinct eigenvalues, that is

$$\bigcup_{k=0}^{\left\lfloor \frac{|n|}{2} \right\rfloor} \mathcal{D}_k^+$$

where

$$\mathcal{D}_k^+ := \left\{ U \mathrm{diag}\left( e^{\frac{i2\pi k}{|n|}}, e^{\frac{-i2\pi k}{|n|}} \right) U^{-1} \,:\, U \in \mathrm{SU}(2) \right\}$$

Now, recall that the single element class $\{I\}$ is connected and that every conjugacy class of a non-central element in SU(2) is connected and homeomorphic to $S^2$. Therefore we know that

$$\mathcal{D}_k^+ \quad \text{is connected for every} \quad k = 0, 1, \ldots, \left\lfloor \frac{|n|}{2} \right\rfloor$$

However, it is important to note that

$$\bigcup_{k=0}^{\left\lfloor\frac{|n|}{2}\right\rfloor} \mathcal{D}_k^+$$

is a disjoint union, thus consists of $\left\lfloor\frac{|n|}{2}\right\rfloor + 1$ connected components. Therefore

$$\bigcup_{k=0}^{\left\lfloor\frac{|n|}{2}\right\rfloor} \mathcal{D}_k^+ \cong \{I\} \sqcup \underbrace{S^2 \sqcup \cdots \sqcup S^2}_{\left\lfloor\frac{|n|}{2}\right\rfloor \text{ times}}$$

Hence, $\mathcal{A}_+^n$ has $\left\lfloor\frac{|n|}{2}\right\rfloor + 1$ connected components.

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□

We have now successfully classified the connectedness of two of our three preimages. Thus we turn our attention to the final preimage which in part partitions our fixed point set. To do so, first we will consider our fixed point stratum sets which arise in the case where the $n$-th power of $A_1$ is in the center of SU(2), specifically when it is equal to minus the identity.

**Lemma 14.** *For every $n \in \mathbb{Z} \setminus \{0\}$ and $A_1 \in \mathcal{A}_-^n$, $\{A_1\} \times \mathcal{F}_{A_1}^-$ has*

$$\left\lfloor\frac{|n|-1}{2}\right\rfloor + 1$$

*connected components*

*Proof.* The proof follows very similarly to that of the $\mathcal{F}_{A_1}^+$ case. Therefore, again fix $n \in \mathbb{Z} \setminus \{0\}$ and fix $A_1 \in \mathcal{A}_-^n$. With

$$\{A_1\} \times \mathcal{F}_{A_1}^- := \{A_1\} \times \left\{(B_1, A_2, B_2, A_3, B_3) \in \mathrm{SU}(2)^5 \, : \, \prod_{i=1}^{3} [A_i, B_i] = I, \; \big([A_3, B_3]A_1\big)^n = -I\right\}$$

we begin by noting that the singleton set $\{A_1\}$ is connected in SU(2) and the product

$$\{A_1\} \times \mathcal{F}_{A_1}^-$$

is homeomorphic to $\mathcal{F}_{A_1}^-$ as it is just a copy of this set at $A_1$. Thus to determine the connectedness of the fixed point stratum set, it suffices to show the connectedness of $\mathcal{F}_{A_1}^-$. With this, we first define

$$\mathcal{B}_{A_1}^- := \big\{(B_1, A_3, B_3) \in \mathrm{SU}(2)^4 \, : \, ([A_3, B_3]A_1)^n = -I\big\}$$

Now consider the map

$$\pi : \mathcal{F}_{A_1}^- \longrightarrow \mathcal{B}_{A_1}^-$$

given by

$$\big(B_1, A_2, B_2, A_3, B_3\big) \longmapsto \big(B_1, A_3, B_3\big)$$

Note that $\pi$ is a restriction of the projection map

$$\mathrm{p} : \mathrm{SU}(2)^5 \longrightarrow \mathrm{SU}(2)^3$$

which we know to be continuous as projections in the product topology are continuous, thus $\pi$ is continuous. Additionally, we claim that $\pi$ is surjective. To see this, take $\big(B_1, A_3, B_3\big) \in \mathcal{B}^-_{A_1}$. Then by the definition of $\mathcal{B}^-_{A_1}$, we have

$$\big([A_3, B_3]A_1\big)^n = -I$$

Thus our goal is to find an $A_2, B_2 \in \mathrm{SU}(2)$ such that the 5-tuple $\big(B_1, A_2, B_2, A_3, B_3\big)$ lies in $\mathcal{F}^-_{A_1}$, that is

$$\big[A_2, B_2\big] = \big[A_1, B_1\big]^{-1}\big[A_3, B_3\big]^{-1}$$

Recall that every element in SU(2) can be expressed as a commutator. Hence, because

$$\big[A_1, B_1\big]^{-1}\big[A_3, B_3\big]^{-1} \in \mathrm{SU}(2)$$

then by the surjectivity of the commutator map in SU(2), there exists an $X$, $Y \in \mathrm{SU}(2)$ such that

$$\big[X, Y\big] = \big[A_1, B_1\big]^{-1}\big[A_3, B_3\big]^{-1}$$

Define

$$A_2 := X \quad \text{and} \quad B_2 := Y.$$

Now, observe that

$$\begin{aligned} &\big[A_1, B_1\big]\big[X, Y\big]\big[A_3, B_3\big] \\ &= \big[A_1, B_1\big]\big(\big[A_1, B_1\big]^{-1}\big[A_3, B_3\big]^{-1}\big)\big[A_3, B_3\big] \\ &= \big(\big[A_1, B_1\big]\big[A_1, B_1\big]^{-1}\big)\big(\big[A_3, B_3\big]^{-1}\big[A_3, B_3\big]\big) \\ &= I \end{aligned}$$

Hence along with the relation in $\mathcal{B}^-_{A_1}$, it follows that

$$\big(B_1, A_2, B_2, A_3, B_3\big) \in \mathcal{F}^-_{A_1}$$

and thus

$$\pi\big(B_1, A_2, B_2, A_3, B_3\big) = \big(B_1, A_3, B_3\big)$$

so $\pi$ is surjective. Now that we have established that $\pi$ is a continuous surjection, let us consider the fibers of this map. Given a point $\big(B_1, A_3, B_3\big) \in \mathcal{B}^-_{A_1}$, the fiber over this point is defined as

$$\pi^{-1}\big(B_1, A_3, B_3\big) = \big\{(B_1, A_2, B_2, A_3, B_3) \in \mathcal{F}^-_{A_1}\big\}$$

Note that since in $\mathcal{F}^-_{A_1}$ the only relation involving $\big(A_2, B_2\big)$ is the product of the commutators, which we can express as

$$\big[A_2, B_2\big] = \big[A_1, B_1\big]^{-1}\big[A_3, B_3\big]^{-1}$$

we may fix $\big(B_1, A_3, B_3\big) \in \mathcal{B}^-_{A_1}$ and define the map

$$\psi : \pi^{-1}\big(B_1, A_3, B_3\big) \longrightarrow \big\{(A_2, B_2) \in \mathrm{SU}(2)^2 \,:\, [A_2, B_2] = [A_1, B_1]^{-1}[A_3, B_3]^{-1}\big\}$$

given by

$$\big(B_1, A_2, B_2, A_3, B_3\big) \longmapsto \big(A_2, B_2\big)$$

Since we fixed our fiber, acting as the domain, by the uniqueness of inverses in SU(2) our map $\psi$ is well-defined and surjective. Moreover, by this uniqueness property and the fact that we fixed our $B_1, A_3$ and $B_3$ arguments, we must have that the map is injective, thus $\psi$ is a bijection. Now we observe that $\psi$ is a restriction of the projection map

$$\mathrm{p} : \mathrm{SU}(2)^5 \longrightarrow \mathrm{SU}(2)^2$$

which we know to be continuous as projections in the product topology are continuous, thus $\psi$ is continuous. Furthermore, $\psi^{-1}$ is a restriction of the inclusion map

$$\mathfrak{i} : \mathrm{SU}(2)^2 \longrightarrow \mathrm{SU}(2)^5$$

which we know to be continuous as inclusions in the product topology are continuous, thus $\psi^{-1}$ is continuous. Therefore, $\psi$ is a homeomorphism, that is

$$\pi^{-1}\big(B_1, A_3, B_3\big) \cong \big\{(A_2, B_2) \in \mathrm{SU}(2)^2 \,:\, [A_2, B_2] = [A_1, B_1]^{-1}[A_3, B_3]^{-1}\big\}$$

Importantly, observe that

$$\big\{(A_2, B_2) \in \mathrm{SU}(2)^2 \,:\, [A_2, B_2] = [A_1, B_1]^{-1}[A_3, B_3]^{-1}\big\}$$

is just a fiber over the commutator map of SU(2) and thus is connected. Since we have shown that an arbitrary fiber, $\pi^{-1}(B_1, A_3, B_3)$, is connected, it follows that the fibers of our map $\pi$ are connected. Therefore, $\pi$ is a continuous, surjective map with connected fibers from our total space $\mathcal{F}^-_{A_1}$ into our defined base space $\mathcal{B}^-_{A_1}$. Thus, we will now investigate the connectedness of this base space $\mathcal{B}^-_{A_1}$. To do so consider the $n$-th power map of SU(2)

$$\mathrm{p}_n : \mathrm{SU}(2) \longrightarrow \mathrm{SU}(2)$$

given by

$$Z \longmapsto Z^n$$

Let us examine the fiber over minus the identity

$$\mathrm{p}_n^{-1}(-I) = \big\{Y \in \mathrm{SU}(2) \,:\, (Y)^n = -I\big\}$$

Note that for any $Y \in \mathrm{p}_n^{-1}(-I)$, $Y$ is a matrix in SU(2), so it is diagonalizable and can be written up to conjugation as

$$Y \sim \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}, \quad \theta \in [0, \pi]$$

Now this represents the conjugacy class determined by the eigenvalues $(e^{i\theta}, e^{-i\theta})$, which by our condition on $Y$ that

$$\big(Y\big)^n = -I$$

must satisfy

$$\big(e^{i\theta}\big)^n = -1 \implies e^{in\theta} = -1 \iff n\theta \equiv \pi \ (\mathrm{mod}\ 2\pi)$$

Therefore, we have

$$\theta = \frac{\pi + 2\pi k}{n} = \frac{(2k+1)\pi}{n}, \quad k \in \mathbb{Z}$$

However, due to the equivalence under conjugation of $\theta \sim -\phi$ and $\theta \sim \theta + 2\pi$ in SU(2), we only consider $\theta \in [0, \pi]$. Thus,

$$0 \leq \frac{(2k+1)\pi}{|n|} \leq \pi \implies 0 \leq k \leq \left\lfloor \frac{|n|-1}{2} \right\rfloor$$

and so $k = 0, 1, \ldots, \left\lfloor \frac{|n|-1}{2} \right\rfloor$. Hence our possible eigenvalue pairs are

$$\left\{ e^{\frac{i(2k+1)\pi}{|n|}}, e^{\frac{-i(2k+1)\pi}{|n|}} \right\} \quad \text{where} \quad k = 0, 1, \ldots, \left\lfloor \tfrac{|n|-1}{2} \right\rfloor$$

Since in SU(2), two matrices are conjugate if and only if they have the same eigenvalues, these pairs form distinct conjugacy classes. So this fiber is equal to the union of the conjugacy classes of the diagonal matrices with these distinct eigenvalues, that is

$$\bigcup_{k=0}^{\left\lfloor \frac{|n|-1}{2} \right\rfloor} \mathcal{D}^-_k$$

where

$$\mathcal{D}^-_k := \left\{ V \mathrm{diag}\left( e^{\frac{i(2k+1)\pi}{|n|}}, e^{\frac{-i(2k+1)\pi}{|n|}} \right) V^{-1} \,:\, V \in \mathrm{SU}(2) \right\}$$

Now, from our relation we know that $[A_3, B_3]A_1$ lies in this fiber of the $n$-th power map, thus

$$[A_3, B_3]A_1 \in \bigcup_{k=0}^{\left\lfloor \frac{|n|-1}{2} \right\rfloor} \mathcal{D}_k^-$$

Recall that the single element class $\{-I\}$ is connected and that every conjugacy class of a non-central element in SU(2) is connected and homeomorphic to $S^2$. Therefore we know that

$$\mathcal{D}_k^- \quad \text{is connected for every} \quad k = 0, 1, \ldots, \left\lfloor \frac{|n|-1}{2} \right\rfloor$$

However, it is important to note that

$$\bigcup_{k=0}^{\left\lfloor \frac{|n|-1}{2} \right\rfloor} \mathcal{D}_k^-$$

is a disjoint union, thus consists of $\left\lfloor \frac{|n|-1}{2} \right\rfloor + 1$ connected components. Therefore

$$\bigcup_{k=0}^{\left\lfloor \frac{|n|-1}{2} \right\rfloor} \mathcal{D}_k^- \cong \{I\} \sqcup \underbrace{S^2 \sqcup \cdots \sqcup S^2}_{\left\lfloor \frac{|n|-1}{2} \right\rfloor \text{ times}}$$

Now note that $B_1$ is not involved in the relation in $\mathcal{B}_{A_1}^-$ and so it is unconstrained in SU(2). We do have, however, that $A_3$ and $B_3$ are involved this relation, thus we will consider the map

$$\Lambda : \mathrm{SU}(2)^2 \longrightarrow \mathrm{SU}(2)$$

given by

$$\big(A_3, B_3\big) \longmapsto \big[A_3, B_3\big] A_1$$

Observe that this is a continuous map as it is defined on a group operation, multiplication, which is smooth in SU(2). The key observation is that by our relation on $\mathcal{B}_{A_1}^-$ we have that

$$\big(B_1, A_3, B_3\big) \in \mathcal{B}_{A_1}^- \iff \Lambda\big(A_3, B_3\big) \in \mathrm{p}_n^{-1}(-I)$$

Therefore,

$$\mathcal{B}_{A_1}^- = \mathrm{SU}(2) \times \Lambda^{-1}\big(\mathrm{p}_n^{-1}(-I)\big)$$

and hence we can express $\mathcal{B}_{A_1}^-$ as the disjoint union of $\left\lfloor \frac{|n|-1}{2} \right\rfloor + 1$ subsets, that is

$$\mathcal{B}_{A_1}^- = \mathrm{SU}(2) \times \left( \bigcup_{k=0}^{\left\lfloor \frac{|n|-1}{2} \right\rfloor} \Big[\Lambda^{-1}\big(\mathcal{D}_k^-\big)\Big] \right)$$

Since $\Lambda$ is continuous and the continuous preimage of a connected set is connected, it follows that

$$\Lambda^{-1}\big(\mathcal{D}_k^-\big)$$

is connected for each $k = 0, 1, \ldots, \left\lfloor \frac{|n|-1}{2} \right\rfloor$. Moreover, since SU(2) is connected and the finite product of a connected spaces is connected, it follows that for each $k = 0, 1, \ldots, \left\lfloor \frac{|n|-1}{2} \right\rfloor$,

$$\mathrm{SU}(2) \times \Lambda^{-1}\big(\mathcal{D}_k^-\big)$$

is connected. Therefore, our set $\mathcal{B}_{A_1}^-$ has $\left\lfloor \frac{|n|-1}{2} \right\rfloor + 1$ connected components. So we have shown that there is a continuous surjection

$$\pi : \mathcal{F}_{A_1}^- \longrightarrow \mathcal{B}_{A_1}^-$$

with non-empty, connected fibers. Since every fiber is non-empty and connected, and the base space $\mathcal{B}^-_{A_1}$ has $\left\lfloor \frac{|n|-1}{2} \right\rfloor + 1$ connected components, it follows that $\mathcal{F}^-_{A_1}$ has $\left\lfloor \frac{|n|-1}{2} \right\rfloor + 1$ connected components. Hence the entire fixed point stratum set

$$\{A_1\} \times \mathcal{F}^-_{A_1}$$

has $\left\lfloor \frac{|n|-1}{2} \right\rfloor + 1$ connected components. □

Having classified the connectedness of each individual fixed point stratum set corresponding to the fibers of $A_1 \in \mathcal{A}^n_-$ under $\mu$, we now wish to consider the union so that we know the number of connected components of the entire preimage.

**Lemma 15.** *For every $n \in \mathbb{Z} \setminus \{0\}$,*

$$\mu^{-1}\big(\mathcal{A}^n_-\big) = \bigcup_{A_1 \in \mathcal{A}^n_-} \{A_1\} \times \mathcal{F}^-_{A_1}$$

*has*

$$\left(\left\lfloor \frac{|n|-1}{2} \right\rfloor + 1\right)^2$$

*connected components.*

*Proof.* Fix $n \in \mathbb{Z} \setminus \{0\}$. We begin by noting that by the previous lemma, we know that for each $A_1 \in \mathcal{A}^n_-$ our corresponding fixed point stratum set

$$\{A_1\} \times \mathcal{F}^-_{A_1} := \{A_1\} \times \left\{ (B_1, A_2, B_2, A_3, B_3) \in \mathrm{SU}(2)^5 \,:\, \prod_{i=1}^{3} \big[A_i, B_i\big] = I,\ \big([A_3, B_3]A_1\big)^n = -I \right\}$$

has $\left\lfloor \frac{|n|-1}{2} \right\rfloor + 1$ connected components. Thus the remaining determination of the connectedness of our preiamge

$$\mu^{-1}\big(\mathcal{A}^n_-\big) = \bigcup_{A_1 \in \mathcal{A}^n_-} \{A_1\} \times \mathcal{F}^-_{A_1}$$

depends on that of the parameter space, our characterization set $\mathcal{A}^n_-$. Now with

$$\mathcal{A}^n_- = \big\{A_1 \in \mathrm{SU}(2) \,:\, \big(A_1\big)^n = -I\big\}$$

let us recall that this is merely the fiber of the identity under the $n$-th power map of SU(2). Thus we will consider this fiber, first defining the $n$-th power map,

$$\mathrm{p}_n : \mathrm{SU}(2) \longrightarrow \mathrm{SU}(2)$$

given by

$$Z \longmapsto Z^n$$

Now the fiber of minus the identity under this map is simply

$$\mathrm{p}_n^{-1}(-I) = \big\{Y \in \mathrm{SU}(2) \,:\, (Y)^n = -I\big\}$$

Now, for any $Y \in \mathrm{p}_n^{-1}(I)$, $Y$ is a matrix in SU(2), so it is diagonalizable and can be written up to conjugation as

$$Y \sim \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}, \quad \theta \in [0, \pi]$$

Now this represents the conjugacy class determined by the eigenvalues $(e^{i\theta}, e^{-i\theta})$, which by our condition on $Y$ that

$$\big(Y\big)^n = -I$$

must satisfy

$$\big(e^{i\theta}\big)^n = -1 \implies e^{in\theta} = -1 \iff n\theta \equiv \pi \pmod{2\pi}$$

Therefore, we have

$$\theta = \frac{\pi + 2\pi k}{n} = \frac{(2k+1)\pi}{n}, \quad k \in \mathbb{Z}$$

However, due to the equivalence under conjugation of $\theta \sim -\theta$ and $\theta \sim \theta + 2\pi$ in SU(2), we only consider $\theta \in [0, \pi]$. Thus,

$$0 \leq \frac{(2k+1)\pi}{|n|} \leq \pi \implies 0 \leq k \leq \left\lfloor \frac{|n|-1}{2} \right\rfloor$$

and so $k = 0, 1, \dots, \left\lfloor \frac{|n|-1}{2} \right\rfloor$. Hence our possible eigenvalue pairs are

$$\left\{ e^{\frac{i(2k+1)\pi}{|n|}}, e^{\frac{-i(2k+1)\pi}{|n|}} \right\} \quad \text{where} \quad k = 0, 1, \dots, \left\lfloor \frac{|n|-1}{2} \right\rfloor$$

Since in SU(2), two matrices are conjugate if and only if they have the same eigenvalues, these pairs form distinct conjugacy classes. So this fiber is equal to the union of the conjugacy classes of the diagonal matrices with these distinct eigenvalues, that is

$$\bigcup_{k=0}^{\left\lfloor \frac{|n|-1}{2} \right\rfloor} \mathcal{D}_k^-$$

where

$$\mathcal{D}_k^- := \left\{ V \mathrm{diag}\left( e^{\frac{i(2k+1)\pi}{|n|}}, e^{\frac{-i(2k+1)\pi}{|n|}} \right) V^{-1} \,:\, V \in \mathrm{SU}(2) \right\}$$

Now, recall that the single element class $\{-I\}$ is connected and that every conjugacy class of a non-central element in SU(2) is connected and homeomorphic to $S^2$. Therefore we know that

$$\mathcal{D}_k^- \quad \text{is connected for every} \quad k = 0, 1, \dots, \left\lfloor \frac{|n|-1}{2} \right\rfloor$$

However, it is important to note that

$$\bigcup_{k=0}^{\left\lfloor \frac{|n|-1}{2} \right\rfloor} \mathcal{D}_k^-$$

is a disjoint union, thus consists of $\left\lfloor \frac{|n|-1}{2} \right\rfloor + 1$ connected components. Therefore

$$\bigcup_{k=0}^{\left\lfloor \frac{|n|-1}{2} \right\rfloor} \mathcal{D}_k^- \cong \{I\} \sqcup \underbrace{S^2 \sqcup \cdots \sqcup S^2}_{\left\lfloor \frac{|n|-1}{2} \right\rfloor \text{ times}}$$

Hence, $\mathcal{A}_-^n$ has $\left\lfloor \frac{|n|-1}{2} \right\rfloor + 1$ connected components.

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□

Having classified the connectedness of the three preimages which make up our fixed point set, we now turn our attention to a few lemmas which will aid in our proof of our main result.

**Lemma 16.** *For every $n \in \mathbb{Z} \setminus \{0\}$, the preimage*

$$\mu^{-1}(\mathcal{A}_+^n) = \bigcup_{A_1 \in \mathcal{A}_+^n} \{A_1\} \times \mathcal{F}_{A_1}^+$$

*has*

$$\left\lfloor \frac{|n|}{2} \right\rfloor + 1$$

*connected components in which $A_3$ and $B_3$ commute.*

*Proof.* We begin by noting that it suffices to show that for each fixed point stratum set $\{A_1\} \times \mathcal{F}_{A_1}^+$, in only one of its

$$\left\lfloor \frac{|n|}{2} \right\rfloor + 1$$

connected components can $A_3$ and $B_3$ commute. It would then follow that in taking the union over our parameter space, the characterization set $\mathcal{A}_+^n$, which in a previous lemma we showed has

$$\left\lfloor \frac{|n|}{2} \right\rfloor + 1$$

connected components, the preimage $\mu^{-1}(\mathcal{A}_+^n)$, would have

$$\left\lfloor \frac{|n|}{2} \right\rfloor + 1$$

connected components were $A_3$ and $B_3$ commute. Thus first we fix $A_1 \in \mathcal{A}_+^n$. Now recall that the

$$\left\lfloor \frac{|n|}{2} \right\rfloor + 1$$

connected components which arise from this fixed point stratum set

$$\{A_1\} \times \mathcal{F}_{A_1}^+ = \{A_1\} \times \left\{ (B_1, A_2, B_2, A_3, B_3) \in \mathrm{SU}(2)^5 \,:\, \prod_{i=1}^{3} [A_i, B_i] = I,\ \big([A_3, B_3]A_1\big)^n = I \right\}$$

are parameterized by that second relation, which corresponds to the stratification of the elements of SU(2) whose $n$-th power is equal to the identity, according to their conjugacy classes. We saw this in the proof of a previous lemma that

$$[A_3, B_3]A_1 \in \bigcup_{k=0}^{\left\lfloor \frac{|n|}{2} \right\rfloor} \mathcal{D}_k^+$$

where

$$\mathcal{D}_k^+ := \left\{ U \mathrm{diag}\left( e^{\frac{i2\pi k}{|n|}}, e^{\frac{-i2\pi k}{|n|}} \right) U^{-1} \,:\, U \in \mathrm{SU}(2) \right\}$$

By assumption $(A_1)^n = I$, thus there are two cases to consider when determining when our $A_3, B_3$ commutator equals the identity. The first case is when $A_1 = \pm I$. Note that this specific case is dependent on the parity of $n$, however, due to the equivalence of $X \sim -X$ under conjugation, for $X \in \mathrm{SU}(2)$, our argument is unaffected by this detail. Thus in this instance our relation becomes

$$\big([A_3, B_3]A_1\big)^n = \big(\pm [A_3, B_3]\big)^n = I$$

and so our commutator lies in the disjoint union of the conjugacy classes, that is

$$\pm [A_3, B_3] \in \bigcup_{k=0}^{\left\lfloor \frac{|n|}{2} \right\rfloor} \mathcal{D}_k^+$$

However, if this is the case then we know that there is only one conjugacy class which contains the identity, namely the singleton conjugacy class $\{I\}$. Thus we must have that only in the connected component corresponding to this conjugacy class can $A_3$ and $B_3$ commute. Next in the case where $A_1 \neq \pm I$, we leverage our assumption that $(A_1)^n = I$ to observe that there exists an $\ell = 1, 2, \ldots, \left\lfloor \frac{|n|}{2} \right\rfloor$ such that

$$A_1 \in \mathcal{D}_\ell^+ := \left\{ Z \mathrm{diag}\left( e^{\frac{i2\pi k}{|n|}}, e^{\frac{-i2\pi k}{|n|}} \right) Z^{-1} \,:\, Z \in \mathrm{SU}(2) \right\}$$

Note that as we previously mentioned, by our relation

$$\big[A_3, B_3\big] A_1 \in \bigcup_{k=0}^{\left\lfloor \frac{|n|}{2} \right\rfloor} \mathcal{D}_k^+$$

however, when $A_3$ and $B_3$ commute

$$\big[A_3, B_3\big] A_1$$

simply becomes $A_1$, which we know lies in the distinct conjugacy class $\mathcal{D}_\ell^+$. Since our conjugacy classes are disjoint, $A_1$ cannot lie in any of the other conjugacy classes and so only in $\mathcal{D}_\ell^+$ can $A_3$ and $B_3$ commute. Therefore we must have that only in the corresponding connected component of this conjugacy class, $A_3$ and $B_3$ commute. Hence we are done. □

Next we will prove the analogous result for $\mu^{-1}\big(\mathcal{A}_-^n\big)$.

**Lemma 17.** *For every $n \in \mathbb{Z} \setminus \{0\}$, the preimage*

$$\mu^{-1}\big(\mathcal{A}_-^n\big) = \bigcup_{A_1 \in \mathcal{A}_-^n} \{A_1\} \times \mathcal{F}_{A_1}^-$$

*has*

$$\left\lfloor \frac{|n|-1}{2} \right\rfloor + 1$$

*connected components in which $A_3$ and $B_3$ commute.*

*Proof.* We begin by noting that it suffices to show that for each fixed point stratum set $\{A_1\} \times \mathcal{F}_{A_1}^-$, in only one of its

$$\left\lfloor \frac{|n|-1}{2} \right\rfloor + 1$$

connected components can $A_3$ and $B_3$ commute. It would then follow that in taking the union over our parameter space, the characterization set $\mathcal{A}_-^n$, which in a previous lemma we showed has

$$\left\lfloor \frac{|n|-1}{2} \right\rfloor + 1$$

connected components, the preimage $\mu^{-1}\big(\mathcal{A}_-^n\big)$, would have

$$\left\lfloor \frac{|n|-1}{2} \right\rfloor + 1$$

connected components were $A_3$ and $B_3$ commute. Thus first we fix $A_1 \in \mathcal{A}_-^n$. Now recall that the

$$\left\lfloor \frac{|n|-1}{2} \right\rfloor + 1$$

connected components which arise from this fixed point stratum set

$$\{A_1\} \times \mathcal{F}_{A_1}^- = \big\{A_1\big\} \times \left\{ (B_1, A_2, B_2, A_3, B_3) \in \mathrm{SU}(2)^5 \,:\, \prod_{i=1}^{3} \big[A_i, B_i\big] = I,\ \big([A_3, B_3]A_1\big)^n = -I \right\}$$

are parameterized by that second relation, which corresponds to the stratification of the elements of SU(2) whose $n$-th power is equal to the identity, according to their conjugacy classes. We saw this in the proof of a previous lemma that

$$[A_3, B_3]A_1 \in \bigcup_{k=0}^{\left\lfloor \frac{|n|-1}{2} \right\rfloor} \mathcal{D}_k^-$$

where

$$\mathcal{D}_k^- := \left\{ V\mathrm{diag}\left( e^{\frac{i(2k+1)\pi}{|n|}}, e^{\frac{-i(2k+1)\pi}{|n|}} \right) V^{-1} \,:\, V \in \mathrm{SU}(2) \right\}$$

By assumption $(A_1)^n = -I$, thus there are two cases to consider when determining when our $A_3, B_3$ commutator equals the identity. The first case is when $A_1 = -I$. Note that this specific case is dependent on the parity of $n$, however, due to the equivalence of $X \sim -X$ under conjugation, for $X \in \mathrm{SU}(2)$, our argument is unaffected by this detail. Thus in this instance our relation becomes

$$\big([A_3, B_3]A_1\big)^n = \big( -[A_3, B_3]\big)^n = -I$$

and so our commutator lies in the disjoint union of the conjugacy classes, that is

$$-[A_3, B_3] \in \bigcup_{k=0}^{\left\lfloor \frac{|n|-1}{2} \right\rfloor} \mathcal{D}_k^-$$

However, if this is the case then we know that there is only one conjugacy class which contains the identity, namely the singleton conjugacy class $\{-I\}$. Thus we must have that only in the connected component corresponding to this conjugacy class can $A_3$ and $B_3$ commute. Next in the case where $A_1 \neq -I$, we leverage our assumption that $(A_1)^n = -I$ to observe that there exists an $\ell = 1, 2, \ldots, \left\lfloor \frac{|n|-1}{2} \right\rfloor$ such that

$$A_1 \in \mathcal{D}_\ell^- := \left\{ Z\mathrm{diag}\left( e^{\frac{i(2k+1)\pi}{|n|}}, e^{\frac{-i(2k+1)\pi}{|n|}} \right) Z^{-1} \,:\, Z \in \mathrm{SU}(2) \right\}$$

Note that as we previously mentioned, by our relation

$$[A_3, B_3]A_1 \in \bigcup_{k=0}^{\left\lfloor \frac{|n|-1}{2} \right\rfloor} \mathcal{D}_k^-$$

however, when $A_3$ and $B_3$ commute

$$[A_3, B_3]A_1$$

simply becomes $A_1$, which we know lies in the distinct conjugacy class $\mathcal{D}_\ell^-$. Since our conjugacy classes are disjoint, $A_1$ cannot lie in any of the other conjugacy classes and so only in $\mathcal{D}_\ell^-$ can $A_3$ and $B_3$ commute. Therefore we must have that only in the corresponding connected component of this conjugacy class, $A_3$ and $B_3$ commute. Hence we are done. □

We are now ready to state and prove our main result about the number of connected components of our fixed point set.

**Theorem 1.** *The fixed point set of the $n$-th power of $\Phi^*$, has*

$$\left\lfloor \frac{n^2}{2} \right\rfloor + 1$$

*connected components.*

*Proof.* Fix $n \in \mathbb{Z} \setminus \{0\}$. With

$$\operatorname{Fix}\big((\Phi^n)^*\big) = \left\{ (A_i, B_i) \in \mathrm{SU}(2)^6 \,:\, \prod_{i=1}^{3} [A_i, B_i] = I,\ \big([A_3, B_3]A_1\big)^n = \big(A_1\big)^n,\ \big[(A_1)^n, A_3\big] = I,\ \big[(A_1)^n, B_3\big] = I \right\}$$

we begin by recalling that

$$\operatorname{Fix}\big((\Phi^n)^*\big) = \mu^{-1}\big(\mathcal{A}^n_{\not\pm}\big) \,\cup\, \big(\mu^{-1}\big(\mathcal{A}^n_{+}\big) \,\cup\, \mu^{-1}\big(\mathcal{A}^n_{-}\big)\big)$$

which is equivalent to

$$\operatorname{Fix}\big((\Phi^n)^*\big) = \bigcup_{A_1 \in \mathcal{A}^n_{\not\pm}} \{A_1\} \times \mathcal{F}^{\not\pm}_{A_1} \,\cup\, \bigcup_{A_1' \in \mathcal{A}^n_{+}} \{A_1'\} \times \mathcal{F}^{+}_{A_1'} \,\cup\, \bigcup_{A_1'' \in \mathcal{A}^n_{-}} \{A_1''\} \times \mathcal{F}^{-}_{A_1''}$$

Now, by a previous lemma we know that the preimage over all $A_1$ whose $n$-th power is non-central in SU(2),

$$\mu^{-1}\big(\mathcal{A}^n_{\not\pm}\big) = \bigcup_{A_1 \in \mathcal{A}^n_{\not\pm}} \{A_1\} \times \mathcal{F}^{\not\pm}_{A_1}$$

has $|n|$ connected components. By a previous lemma we know that the preimage over all $A_1$ whose $n$-th power is equal to the identity,

$$\mu^{-1}\big(\mathcal{A}^n_{+}\big) = \bigcup_{A_1' \in \mathcal{A}^n_{+}} \{A_1'\} \times \mathcal{F}^{+}_{A_1'}$$

has

$$\left( \left\lfloor \frac{|n|}{2} \right\rfloor + 1 \right)^2$$

connected components. Finally, by a previous lemma we know that the preimage over all $A_1$ whose $n$-th power is equal to minus the identity,

$$\mu^{-1}\big(\mathcal{A}^n_{-}\big) = \bigcup_{A_1'' \in \mathcal{A}^n_{-}} \{A_1''\} \times \mathcal{F}^{-}_{A_1''}$$

has

$$\left( \left\lfloor \frac{|n|-1}{2} \right\rfloor + 1 \right)^2$$

connected components. With this, we note that

$$\bigcup_{A_1 \in \mathcal{A}^n_{\not\pm}} \{A_1\} \times \mathcal{F}^{\not\pm}_{A_1} \,\cup\, \bigcup_{A_1' \in \mathcal{A}^n_{+}} \{A_1'\} \times \mathcal{F}^{+}_{A_1'} \,\cup\, \bigcup_{A_1'' \in \mathcal{A}^n_{-}} \{A_1''\} \times \mathcal{F}^{-}_{A_1''}$$

is by construction a disjoint union, as necessarily our $A_1$ arguments, which parametrize each individual fixed point stratum set, cannot agree. Therefore, we get an upper bound for the number of connected components of our fixed point set by adding the number of connected components from each respective preimage, that is, we get that our fixed point set has at most

$$|n| + \left( \left\lfloor \frac{|n|}{2} \right\rfloor + 1 \right)^2 + \left( \left\lfloor \frac{|n|-1}{2} \right\rfloor + 1 \right)^2$$

connected components However, this is just an upper bound and in fact we claim that the actual number of connected components for our fixed point set is much lower. To refine this upper bound on the number of connected components of our fixed point set we will leverage the fact that the closure of a connected set is connected and that if two connected sets intersect in their closure, their union is connected. Thus with this it suffices to show that the intersection of the closures of specific connected components, arising from fixed point stratum sets, are non-empty and thus come together to form larger connected components. First, though, we need to identify which of our connect components could potentially intersect in their closures. To do so let us consider our respective fixed point stratum sets for arbitrary, $A_1 \in \mathcal{A}^n_{\not\pm}$, $A_1' \in \mathcal{A}^n_+$, and $A_1'' \in \mathcal{A}^n_-$, as these give rise to our connected components. By definition we have

$$\{A_1\} \times \mathcal{F}^{\not\pm}_{A_1} = \{A_1\} \times \Big\{(B_1, A_2, B_2, A_3, B_3) \in \mathrm{SU}(2)^5 \,:\, \big[A_1, B_1\big] = \big[A_2, B_2\big]^{-1},\ \big[A_3, B_3\big] = \big[(A_1)^n, A_3\big] = \big[(A_1)^n, B_3\big] = I$$

$$\{A_1'\} \times \mathcal{F}^+_{A_1'} = \{A_1'\} \times \left\{(B_1, A_2, B_2, A_3, B_3) \in \mathrm{SU}(2)^5 \,:\, \prod_{i=1}^{3} \big[A_i, B_i\big] = I,\ \big([A_3, B_3]A_1'\big)^n = I\right\}$$

$$\{A_1''\} \times \mathcal{F}^-_{A_1''} = \{A_1''\} \times \left\{(B_1, A_2, B_2, A_3, B_3) \in \mathrm{SU}(2)^5 \,:\, \prod_{i=1}^{3} \big[A_i, B_i\big] = I,\ \big([A_3, B_3]A_1''\big)^n = -I\right\}$$

Now since SU(2) is a compact Lie group, the closure of a set corresponds to the inclusion of all points that can be approximated by sequences in the set. Since the relations on that set are continuous, any limit point will satisfy those relations as well. Thus if the closures of two subsets of SU(2) intersect, the points in the intersection must satisfy the relations of both sets. With this we observe that for each fixed point stratum set in the union

$$\mu^{-1}\big(\mathcal{A}^n_{\not\pm}\big) = \bigcup_{A_1 \in \mathcal{A}^n_{\not\pm}} \{A_1\} \times \mathcal{F}^{\not\pm}_{A_1}$$

we have that $A_3$ and $B_3$ commute. Therefore, if we consider the closures of the corresponding $|n|$ connected components, were the closures of any of the other connected components arising from our other two respective collections of fixed point stratum sets to intersect, they too must satisfy this relation. By two previous lemmas we know that

$$\mu^{-1}\big(\mathcal{A}^n_+\big) = \bigcup_{A_1' \in \mathcal{A}^n_+} \{A_1'\} \times \mathcal{F}^+_{A_1'}$$

has

$$\left\lfloor \frac{|n|}{2} \right\rfloor + 1$$

connected components which satisfy this commutation relation and that

$$\mu^{-1}\big(\mathcal{A}^n_-\big) = \bigcup_{A_1'' \in \mathcal{A}^n_-} \{A_1''\} \times \mathcal{F}^-_{A_1''}$$

has

$$\left\lfloor \frac{|n| - 1}{2} \right\rfloor + 1$$

connected components which satisfy this commutation relation. Moreover, these connected components which correspond with when $A_3$ and $B_3$ commute, will be of the form

$$\{A_1'\} \times \mathcal{F}^+_{A_1'} = \{A_1'\} \times \Big\{(B_1, A_2, B_2, A_3, B_3) \in \mathrm{SU}(2)^5 \,:\, \big[A_1', B_1\big] = \big[A_2, B_2\big]^{-1},\ \big[A_3, B_3\big] = I\Big\}$$

$$\{A_1''\} \times \mathcal{F}^-_{A_1''} = \{A_1''\} \times \Big\{(B_1, A_2, B_2, A_3, B_3) \in \mathrm{SU}(2)^5 \,:\, \big[A_1'', B_1\big] = \big[A_2, B_2\big]^{-1},\ \big[A_3, B_3\big] = I\Big\}$$

respectively. Note that it immediately follows that all relations for any fixed point stratum set $\{A_1\} \times \mathcal{F}^{\not\pm}_{A_1}$ are satisfied, as by assumption, for each respective fixed point stratum set in the latter preimages, we have by construction that $\big(A_1'\big)^n = I$ and $\big(A_1''\big)^n = -I$, thus trivially commute with $A_3$ and $B_3$. It is important to note that these specific fixed point stratum sets decorate the gaps in between the connected components of $\mu^{-1}\big(\mathcal{A}^n_{\not\pm}\big)$ in correspondence to their respective open intervals along $[0, \pi]$ We claim that all of the special connected components of our fixed point stratum sets, that is the ones in which $A_3$ and $B_3$ commute, from our two respective preimages

$$\mu^{-1}\big(\mathcal{A}^n_+\big) = \bigcup_{A_1' \in \mathcal{A}^n_+} \{A_1'\} \times \mathcal{F}^-_{A_1'} \quad \text{and} \quad \mu^{-1}\big(\mathcal{A}^n_-\big) = \bigcup_{A_1'' \in \mathcal{A}^n_-} \{A_1''\} \times \mathcal{F}^-_{A_1''}$$

merge all of the connected components from our third preimage, to form one large connected component. This claim may be verified through the previously mentioned intersection of the closure argument. We will look at one case of this a note that we may repeat the argument to sew together all of our target connected components. For this case, fix $A_1 \in \mathrm{SU}(2)$. Then by construction if $\big(A_1\big)^n = I$ we have that $\mu^{-1}\big(A_1\big) \cong \{A_1\} \times \mathcal{F}^+_{A_1}$. Define the path

$$A_1(t) : \big[0,1\big] \longrightarrow \mu^{-1}\big(\mathrm{SU}(2)\big)$$

where $\big(A_1(0)\big)^n = I$ and $\big(A_1(t)\big)^n \neq \pm I$ for $t \neq 0$. Now observe that

$$\lim_{t\to 0} \mu^{-1}\big(A_1(t)\big) \subseteq \{A_1(0)\} \times \mathcal{F}^+_{A_1}$$

Thus the intersection of the closures of the two sets is non-empty and so their union is connected. As we mentioned this is true for all of our special connected components of our fixed point stratum sets, thus we can connect our

$$\left(\left\lfloor \frac{|n|}{2} \right\rfloor + 1\right) + \left(\left\lfloor \frac{|n|-1}{2} \right\rfloor + 1\right)$$

special connected components in $\mu^{-1}\big(\mathcal{A}^n_{\pm}\big)$ and $|n|$ connected components of $\mu^{-1}\big(\mathcal{A}^n_{\not\pm}\big)$, to form one large connected component. This however is the extent of this merging of connected components that we see from the general fixed point set. This is a result of the following. Suppose that for some fixed $A_1' \in \mathcal{A}^n_+$, and $A_1'' \in \mathcal{A}^n_-$ we had that

$$\overline{\{A_1'\} \times \mathcal{F}^+_{A_1'}} \cap \overline{\{A_1''\} \times \mathcal{F}^-_{A_1''}} \neq \emptyset$$

that is the closures of at least one of the connected components from each of the

$$\left\lfloor \frac{|n|}{2} \right\rfloor + 1 \quad \text{and} \quad \left\lfloor \frac{|n|-1}{2} \right\rfloor + 1$$

respective connected components from the fixed point stratum sets intersected. Then there would exist an element

$$\big(\tilde{A_1}, \tilde{B_1}, \tilde{A_2}, \tilde{B_2}, \tilde{A_3}, \tilde{B_3}\big) \in \overline{\{A_1'\} \times \mathcal{F}^+_{A_1'}} \cap \overline{\{A_1''\} \times \mathcal{F}^-_{A_1''}}$$

such that $\tilde{A_1}, \tilde{B_1}, \tilde{A_2}, \tilde{B_2}, \tilde{A_3}$ and $\tilde{B_3}$ satisfy the relations of each respective set, that is

$$\prod_{i=1}^{3} \big[\tilde{A_i}, \tilde{B_i}\big] = I$$

$$\big([\tilde{A_3}, \tilde{B_3}]\tilde{A_1}\big)^n = I$$

$$\big([\tilde{A_3}, \tilde{B_3}]\tilde{A_1}\big)^n = -I$$

However, in $\mathrm{SU}(2)$ the $n$-th power of a matrix cannot simultaneously be equal to both the identity and minus the identity. Therefore, there cannot exists an element

$$\big(\tilde{A_1}, \tilde{B_1}, \tilde{A_2}, \tilde{B_2}, \tilde{A_3}, \tilde{B_3}\big) \in \overline{\{A_1'\} \times \mathcal{F}^+_{A_1'}} \cap \overline{\{A_1''\} \times \mathcal{F}^-_{A_1''}}$$

and so we are unable to form larger connected components from any of the connected components of each respective fixed point stratum set without the additional components from our other preimage. Therefore the number of connected components in our fixed point set is

$$|n| + \left(\left\lfloor \tfrac{|n|}{2} \right\rfloor + 1\right)^2 + \left(\left\lfloor \tfrac{|n|-1}{2} \right\rfloor + 1\right)^2 - \left(\left\lfloor \tfrac{|n|}{2} \right\rfloor + 1\right) - \left(\left\lfloor \tfrac{|n|-1}{2} \right\rfloor + 1\right) - (|n| - 1)$$

which simplifies to

$$\left\lfloor \frac{n^2}{2} \right\rfloor + 1$$

and so we are done. □

# 5. Character Variety of Dehn Twists about Bounding Pairs

**5.**

## 5.1. Fixed Point Equations for $\big((T_{\gamma_1} \circ T_{\gamma_2}^{-1})^n\big)^*$.

5.1. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .

- On $R(\Sigma, G)$ :

$$\begin{aligned} &A_1 \mapsto \big([A_3,B_3]A_1\big)^n A_1\big(A_1^{-1}[A_3,B_3]^{-1}\big)^n \\ &B_1 \mapsto B_1\big(A_1\big)^n\big(A_1^{-1}[A_3,B_3]^{-1}\big)^n \\ &A_2 \mapsto A_2 \\ &B_2 \mapsto B_2 \\ &A_3 \mapsto \big([A_3,B_3]A_1\big)^n A_3 \big(A_1^{-1}[A_3,B_3]^{-1}\big)^n \\ &B_3 \mapsto \big([A_3,B_3]A_1\big)^n B_3 \big(A_1^{-1}[A_3,B_3]^{-1}\big)^n \end{aligned}$$

- On $R(\Sigma, G)/G$ :

$$\begin{aligned} &T^{-1}A_1T \mapsto \big([A_3,B_3]A_1\big)^n A_1\big(A_1^{-1}[A_3,B_3]^{-1}\big)^n \\ &T^{-1}B_1T \mapsto B_1\big(A_1\big)^n\big(A_1^{-1}[A_3,B_3]^{-1}\big)^n \\ &T^{-1}A_2T \mapsto A_2 \\ &T^{-1}B_2T \mapsto B_2 \\ &T^{-1}A_3T \mapsto \big([A_3,B_3]A_1\big)^n A_3 \big(A_1^{-1}[A_3,B_3]^{-1}\big)^n \\ &T^{-1}B_3T \mapsto \big([A_3,B_3]A_1\big)^n B_3 \big(A_1^{-1}[A_3,B_3]^{-1}\big)^n \end{aligned}$$

## 5.2. Computing $\mathrm{Fix}\big((\Phi^n)^*\big)$.

5.2. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .

- $A_1$ :

$$\begin{gathered} TA_1T^{-1} = \big([A_3,B_3]A_1\big)^n A_1 \big(A_1^{-1}[A_3,B_3]^{-1}\big)^n \\ \big(A_1^{-1}[A_3,B_3]^{-1}\big)^n T\,A_1\,T^{-1}\big([A_3,B_3]A_1\big)^n = A_1 \\ \big(A_1^{-1}[A_3,B_3]^{-1}\big)^n TA_1 = A_1\big(A_1^{-1}[A_3,B_3]^{-1}\big)^n T \\ \big[A_1,\,(A_1^{-1}[A_3,B_3]^{-1})^n T\big] = I \\ \iff (A_1^{-1}[A_3,B_3]^{-1})^n\, T \in C_{\mathrm{SU}(2)}(A_1) \end{gathered}$$

- $B_1$ :

$$\begin{gathered} TB_1T^{-1} = B_1\big(A_1\big)^n\big(A_1^{-1}[A_3,B_3]^{-1}\big)^n \\ B_1^{-1}TB_1T^{-1} = \big(A_1\big)^n\big(A_1^{-1}[A_3,B_3]^{-1}\big)^n \\ [B_1^{-1},T] = \big(A_1\big)^n\big(A_1^{-1}[A_3,B_3]^{-1}\big)^n \\ \big(A_1^{-1}\big)^n[B_1^{-1},T] = \big(A_1^{-1}[A_3,B_3]^{-1}\big)^n \end{gathered}$$

- $A_2$ :

$$\begin{gathered} TA_2T^{-1} = A_2 \\ TA_2 = A_2T \\ \big[A_2,\,T\big] = I \\ \iff T \in C_{\mathrm{SU}(2)}(A_2) \end{gathered}$$

- $B_2$ :

$$TB_2T^{-1} = B_2$$
$$TB_2 = B_2T$$
$$\big[B_2, T\big] = I$$
$$\iff T \in C_{\mathrm{SU}(2)}(B_2)$$

- $A_3$ :

$$TA_3T^{-1} = \big([A_3, B_3]A_1\big)^n A_3 \left(A_1^{-1}[A_3, B_3]^{-1}\right)^n$$
$$\left(A_1^{-1}[A_3, B_3]^{-1}\right)^n T\, A_3\, T^{-1}\big([A_3, B_3]A_1\big)^n = A_3$$
$$\left(A_1^{-1}[A_3, B_3]^{-1}\right)^n T\, A_3 = A_3 \left(A_1^{-1}[A_3, B_3]^{-1}\right)^n T$$
$$\big[A_3, (A_1^{-1}[A_3, B_3]^{-1})^n\, T\big] = I$$
$$\iff \left(A_1^{-1}[A_3, B_3]^{-1}\right)^n T \in C_{\mathrm{SU}(2)}(A_3)$$

- $B_3$ :

$$TB_3T^{-1} = \big([A_3, B_3]A_1\big)^n B_3 \left(A_1^{-1}[A_3, B_3]^{-1}\right)^n$$
$$\left(A_1^{-1}[A_3, B_3]^{-1}\right)^n T\, B_3\, T^{-1}\big([A_3, B_3]A_1\big)^n = B_3$$
$$\left(A_1^{-1}[A_3, B_3]^{-1}\right)^n T\, B_3 = B_3 \left(A_1^{-1}[A_3, B_3]^{-1}\right)^n T$$
$$\big[B_3, (A_1^{-1}[A_3, B_3]^{-1})^n\, T\big] = I$$
$$\iff \left(A_1^{-1}[A_3, B_3]^{-1}\right)^n T \in C_{\mathrm{SU}(2)}(B_3)$$

From these fixed point equations we may observe that our fixed point set of the character variety is partitioned into smaller fixed point stratum sets based on what $T$ is in SU(2). Therefore, we will examine the possible values of $T$ in SU(2). To start if $T$ is in the center of SU(2) then our fixed point equations simplify to those of our representation variety, which we have already classified. Thus let us consider when $T$ is a non-central element of SU(2). Note that by our fixed point equations for $A_1$, $A_3$, and $B_3$ we see that

$$\left(A_1^{-1}[A_3, B_3]^{-1}\right)^n T$$

must be in the centralizer of each respective element. This gives rise to two cases, the first being that

$$\left(A_1^{-1}[A_3, B_3]^{-1}\right)^n T = \pm I$$

and the second that

$$\left(A_1^{-1}[A_3, B_3]^{-1}\right)^n T \neq \pm I$$

In the latter case where $\left(A_1^{-1}[A_3, B_3]^{-1}\right)^n T$ is a non-central element in SU(2), we observe that since it lies in the centralizers of $A_1$, $A_3$, and $B_3$ respectively, then equivalently, $A_1$, $A_3$, and $B_3$ lie in the centralizer of $\left(A_1^{-1}[A_3, B_3]^{-1}\right)^n T$ which in SU(2) we know to be a maximal torus. Recall that in SU(2) every maximal torus is abelian, hence, $A_1$, $A_3$, and $B_3$ must commute. However, if $A_3$ and $B_3$ commute then

$$\big[A_3, B_3\big] = I$$

With this information, let us re-evaluate our fixed point equations. First, for $A_1$ we have

$$TA_1T^{-1} = \big([A_3, B_3]A_1\big)^n A_1 \left(A_1^{-1}[A_3, B_3]^{-1}\right)^n$$
$$TA_1T^{-1} = A_1 \big(A_1\big)^{n-1} A_1 \left(A_1^{-1}\right)^n$$
$$TA_1T^{-1} = A_1$$
$$TA_1 = A_1T$$

Thus we find that $A_1$ commutes with $T$. Now, for $B_1$ we see that

$$TB_1T^{-1} = B_1\big(A_1\big)^n\big(A_1^{-1}[A_3,B_3]^{-1}\big)^n$$
$$TB_1T^{-1} = B_1\big(A_1\big)^n\big(A_1^{-1}\big)^n$$
$$TB_1T^{-1} = B_1$$
$$TB_1 = B_1T$$

Thus again we find that $B_1$ commutes with $T$. Since the fixed point equations for $A_2$ and $B_2$ are unaffected by our new found commutativity, besides the fact that they now commute as our $T$ is assumed to be non-central, we will move on to $A_3$ and $B_3$. First, for $A_3$ observe that

$$TA_3T^{-1} = \big([A_3,B_3]A_1\big)^n A_3 \big(A_1^{-1}[A_3,B_3]^{-1}\big)^n$$
$$TA_3T^{-1} = \big(A_1\big)^n A_3 \big(A_1^{-1}\big)^n$$
$$TA_3T^{-1} = A_3\big(A_1\big)^n\big(A_1^{-1}\big)^n$$
$$TA_3T^{-1} = A_3$$
$$TA_3 = A_3T$$

note that from line 2 to line 3 we leverage the fact that $A_1$ and $A_3$ commute. From this we see that $A_3$ commutes with $T$. Finally, for $B_3$ we observe that in a similar fashion to $A_3$

$$TB_3T^{-1} = \big([A_3,B_3]A_1\big)^n B_3 \big(A_1^{-1}[A_3,B_3]^{-1}\big)^n$$
$$TB_3T^{-1} = \big(A_1\big)^n B_3 \big(A_1^{-1}\big)^n$$
$$TB_3T^{-1} = B_3\big(A_1\big)^n\big(A_1^{-1}\big)^n$$
$$TB_3T^{-1} = B_3$$
$$TB_3 = B_3T$$

Thus we find that $B_3$ commutes with $T$. Now we have shown that every argument commutes with $T$ and by assumption $T$ is non-central in $\mathrm{SU}(2)$ thus $A_1, B_1, A_2, B_2, A_3$ and $B_3$ all lie in the same maximal torus and so they commute. In the case where

$$\big(A_1^{-1}[A_3,B_3]^{-1}\big)^n T = \pm I$$

then we may again re-evaluate our fixed point equations. First, note that the fixed point equations for $A_1$, $A_3$, and $B_3$ all reduce the trivial equation, and so we may disregard them. Next, for $B_1$, we see that

$$TB_1T^{-1} = B_1\big(A_1\big)^n\big(A_1^{-1}[A_3,B_3]^{-1}\big)^n$$
$$TB_1 = B_1\big(A_1\big)^n\big(A_1^{-1}[A_3,B_3]^{-1}\big)^nT$$
$$TB_1 = \pm B_1\big(A_1\big)^n$$
$$B_1^{-1}TB_1 = \pm\big(A_1\big)^n$$

Finally, for $A_2$ and $B_2$ they are unchanged by our new found relation and so we still have that both commute with $T$ and with each other as our $T$ is assumed to be non-central. Therefore, we may express our fixed point set as the union of these three separate cases, that is

$$\mathrm{Fix}\big((\Phi^n)^*\big)_{cv} = \mathcal{F}\big((\Phi^n)^*\big)_{rv} \,\cup\, \mathcal{F}\big((\Phi^n)^*\big)_{\not\equiv} \,\cup\, \mathcal{F}\big((\Phi^n)^*\big)_{\pm}$$

where

$$\mathcal{F}\big((\Phi^n)^*\big)_{rv} = \left\{(A_i,B_i)\in \mathrm{SU}(2)^6 \,:\, \prod_{i=1}^{3}\,[A_i,B_i] = I,\ \big([A_3,B_3]A_1\big)^n = \big(A_1\big)^n,\ \big[(A_1)^n,A_3\big] = I,\ \big[(A_1)^n,B_3\big] = I\right\}$$

our fixed point set of the representation variety,

$$\mathcal{F}\big((\Phi^n)^*\big)_{\not\equiv} = \big\{(A_i,B_i)\in \mathrm{SU}(2)^6 \,:\, \big[A_i,A_j\big] = \big[A_i,B_j\big] = \big[B_i,B_j\big] = I,\ 1\le i,j\le 3\big\}$$

our fixed point set when $T$ is a non-central element of SU(2) and $\left(A_1^{-1}[A_3,B_3]^{-1}\right)^n T \neq I$, and finally

$$\mathcal{F}\big((\Phi^n)^*\big)_{\pm} = \Big\{(A_i,B_i) \in \mathrm{SU}(2)^6 \;:\; \big[A_1,B_1\big] = \big[A_3,B_3\big]^{-1},\ \big[A_2,B_2\big] = I,\ B_1^{-1}TB_1 = \pm\big(A_1\big)^n\Big\}$$

our fixed point set when $T$ is a non-central element of SU(2) and $\left(A_1^{-1}[A_3,B_3]^{-1}\right)^n T = \pm I$. Thus in order to classify the connectedness of our fixed point set for the character variety, it suffices to determine the connectedness of each respective fixed point stratum set in this union. First, as previously noted, $\mathcal{F}\big((\Phi^n)^*\big)_{rv}$ is just a copy of our fixed point set for the representation variety, which we have shown has $\left\lfloor \frac{|n|^2}{2} \right\rfloor + 1$ connected components. Next let us consider our fixed point set when $T$ is a non-central element of SU(2) and $\left(A_1^{-1}[A_3,B_3]^{-1}\right)^n T \neq I$.

**Lemma 18.** *$\mathcal{F}\big((\Phi^n)^*\big)_{\not\pm}$ is connected for every $n \in \mathbb{Z} \setminus \{0\}$*

*Proof.* Fix $n \in \mathbb{Z} \setminus \{0\}$. With

$$\mathcal{F}\big((\Phi^n)^*\big)_{\not\pm} = \big\{(A_i,B_i) \in \mathrm{SU}(2)^6 \;:\; \big[A_i,A_j\big] = \big[A_i,B_j\big] = \big[B_i,B_j\big] = I,\ 1 \le i,j \le 3\big\}$$

we observe that this is just the set of commuting 6-tuples in $\mathrm{SU}(2)^6$. Recall that any two elements of SU(2) commute if and only if they lie in the same maximal torus, which in SU(2) is conjugate to the subgroup of diagonal matrices. Let $\mathrm{M}_T$ denote a maximal torus in SU(2),

$$\mathrm{M}_T = \left\{\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta}\end{pmatrix} : \theta \in [0,2\pi)\right\}$$

Since $A_1, B_2, A_2, B_2, A_3, B_3$ commute, there exists $g \in \mathrm{SU}(2)$ such that

$$g\,A_1g^{-1},\ g\,B_1g^{-1},\ g\,A_2g^{-1},\ g\,B_2g^{-1},\ g\,A_3g^{-1},\ g\,B_3g^{-1} \in \mathrm{M}_T$$

This follows from the fact that all maximal tori in SU(2) are conjugate and every element is contained in some maximal torus. Therefore, we have

$$\begin{aligned}\mathcal{F}\big((\Phi^n)^*\big)_{\not\pm} &= \big\{(A_i,B_i) \in \mathrm{SU}(2)^6 \;:\; \big[A_i,A_j\big] = \big[A_i,B_j\big] = \big[B_i,B_j\big] = I,\ 1 \le i,j \le 3\big\} \\ &= \big\{(g\,t_1g^{-1},\ g\,t_2g^{-1},\ g\,t_3g^{-1},\ g\,t_4g^{-1},\ g\,t_5g^{-1},\ g\,t_6g^{-1}) \;:\; g \in \mathrm{SU}(2),\ t_i \in \mathrm{M}_T\big\}\end{aligned}$$

Now, define the map

$$\Omega : \mathrm{SU}(2) \times \mathrm{M}_T^6 \longrightarrow \mathcal{F}\big((\Phi^n)^*\big)_{\not\pm}$$

by

$$\Omega\big(g,(t_1,t_2,t_3,t_4,t_5,t_6)\big) = \big(g\,t_1g^{-1},\ g\,t_2g^{-1},\ g\,t_3g^{-1},\ g\,t_4g^{-1},\ g\,t_5g^{-1},\ g\,t_6g^{-1}\big).$$

Observe that the domain of the map, $\mathrm{SU}(2) \times \mathrm{M}_T^6$, is connected since SU(2) is connected, $\mathrm{M}_T$ is connected, and the finite product of connected spaces is connected. Therefore, it suffices to show that $\Omega$ is continuous and surjective, as the image of a connected space under a continuous map is connected. We begin by verifying the surjectivity of $\Omega$. Let $(a_1,b_1,a_2,b_2,a_3,b_3) \in \mathcal{F}\big((\Phi^n)^*\big)_{\not\pm}$, then there exists $g \in \mathrm{SU}(2)$ such that

$$g\,a_1g^{-1},\ g\,b_1g^{-1},\ g\,a_2g^{-1},\ g\,b_2g^{-1},\ g\,a_3g^{-1},\ g\,b_3g^{-1} \in \mathrm{M}_T$$

This follows from the fact that all maximal tori in SU(2) are conjugate and every element is contained in some maximal torus. Now, if we set

$$t_1 = g^{-1}a_1g, \quad t_2 = g^{-1}b_1g, \quad t_3 = g^{-1}a_2g \quad t_4 = g^{-1}b_2g, \quad t_5 = g^{-1}a_3g, \quad t_6 = g^{-1}b_3g$$

then by conjugacy of the maximal torus, every element of $\mathcal{F}\big((\Phi^n)^*\big)_{\not\pm}$ lies in the image of $\Omega$ and so the map is surjective. For the continuity of the map, we note that $\Omega$ is defined on group operations, multiplication and inversion, which are smooth in SU(2), and so the map is continuous. Hence, $\mathcal{F}\big((\Phi^n)^*\big)_{\not\pm}$ is connected. □

We have now classified the connectedness of two of our three fixed point stratum sets which partition our fixed point set of the character variety. We now turn our attention to the final fixed point stratum set. However, before we tackle the classification of this set lets first prove an identity that will help us in the proof.

**Lemma 19.** *For any $A_1, B_1 \in \mathrm{SU}(2)$ and $n \in \mathbb{Z} \setminus \{0\}$,* $\big([A_1,B_1]^{-1}A_1\big)^n = B_1\big(A_1\big)^n B_1^{-1}$

*Proof.* We will proceed by induction. First fix $A_1, B_1 \in \mathrm{SU}(2)$ and consider the case where $n = 1$, in this instance we have

$$\begin{aligned}[A_1,B_1]^{-1}A_1 &= B_1A_1B_1^{-1}A_1^{-1}A_1 \\ &= B_1A_1B_1^{-1}\end{aligned}$$

and so our base case holds. Next, suppose that

$$\big([A_1,B_1]^{-1}A_1\big)^n = B_1\big(A_1\big)^n B_1^{-1}$$

for some $n \in \mathbb{Z} \setminus \{0\}$. Then, we want to show that for $n+1$ our relation holds, that is

$$\big([A_1,B_1]^{-1}A_1\big)^{n+1} = B_1\big(A_1\big)^{n+1} B_1^{-1}$$

Now, by our induction hypothesis we have

$$\big([A_1,B_1]^{-1}A_1\big)^n = B_1\big(A_1\big)^n B_1^{-1}$$

and so

$$\begin{aligned}\big([A_1,B_1]^{-1}A_1\big)^{n+1} &= \big([A_1,B_1]^{-1}A_1\big)^n\big([A_1,B_1]^{-1}A_1\big) \\ &= B_1\big(A_1\big)^n B_1^{-1}\big([A_1,B_1]^{-1}A_1\big) \\ &= B_1\big(A_1\big)^n B_1^{-1}B_1A_1B_1^{-1}A_1^{-1}A_1 \\ &= B_1\big(A_1\big)^n A_1B_1^{-1} \\ &= B_1\big(A_1\big)^{n+1}B_1^{-1}\end{aligned}$$

Hence, we are done. □

Now that we have proven this identity we are ready to tackle our final fixed point stratum set.

**Lemma 20.** $\mathcal{F}\big((\Phi^n)^*\big)_\pm$ *is connected for every $n \in \mathbb{Z} \setminus \{0\}$*

*Proof.* Fix $n \in \mathbb{Z} \setminus \{0\}$. With

$$\mathcal{F}\big((\Phi^n)^*\big)_\pm = \Big\{(A_i,B_i) \in \mathrm{SU}(2)^6 \,:\, \big[A_1,B_1\big] = \big[A_3,B_3\big]^{-1},\ \big[A_2,B_2\big] = I,\ B_1^{-1}TB_1 = \pm\big(A_1\big)^n\Big\}$$

we begin by observing that we may expressed our relation

$$B_1^{-1}TB_1 = \pm\big(A_1\big)^n$$

as

$$B_1^{-1}\big([A_1,B_1]^{-1}A_1\big)^n B_1 = \big(A_1\big)^n$$

which by **Lemma 2** is trivial since

$$\begin{aligned}B_1^{-1}\big([A_1,B_1]^{-1}A_1\big)^n B_1 &= \big(A_1\big)^n \\ B_1^{-1}B_1\big(A_1\big)^n B_1^{-1}B_1 &= \big(A_1\big)^n \\ B_1^{-1}B_1\big(A_1\big)^n B_1^{-1}B_1 &= \big(A_1\big)^n \\ \big(A_1\big)^n &= \big(A_1\big)^n\end{aligned}$$

Now the equality of the different relations follows from the assumption that

$$\big(A_1^{-1}[A_3,B_3]^{-1}\big)^n T = \pm I$$

which gives us that

$$T = \pm\big((A_1^{-1}[A_3, B_3]^{-1})^n\big)^{-1} = \pm\big([A_3, B_3]A_1\big)^n$$

However, from our other relations we know that

$$\big[A_3, B_3\big] = \big[A_1, B_1\big]^{-1}$$

and so we may rewrite our $T$ as

$$T = \pm\big([A_1, B_1]^{-1}A_1\big)^n$$

Therefore, our derived relation

$$B_1^{-1}TB_1 = \pm\big(A_1\big)^n$$

becomes

$$B_1^{-1}\big([A_1, B_1]^{-1}A_1\big)^n B_1 = \big(A_1\big)^n$$

where we can drop the $\pm$ sign as the parity of both sides of the equality are determined by the parity of the assumed identity

$$\big(A_1^{-1}[A_3, B_3]^{-1}\big)^n T = \pm I$$

Therefore, our fixed point stratum set can be rewritten as

$$\mathcal{F}\big((\Phi^n)^*\big)_\pm = \Big\{(A_i, B_i) \in \mathrm{SU}(2)^6 \,:\, \big[A_1, B_1\big] = \big[A_3, B_3\big]^{-1},\ \big[A_2, B_2\big] = I\Big\}$$

We now want to show that $\mathcal{F}\big((\Phi^n)^*\big)_\pm$ is connected. We begin by noting that by our relations, we know that $A_1$ and $B_1$ are essentially unconstrained in the fixed point stratum set if we define $A_3$ and $B_3$ based on our choices of $A_1$ and $B_1$. Therefore, we define the commutator map

$$\mathfrak{c} : \mathrm{SU}(2)^2 \longrightarrow \mathrm{SU}(2)$$

given by

$$\big(X, Y\big) \longmapsto \big[X, Y\big]$$

Now if we consider the fiber over $[A_1, B_1]^{-1}$ observe that this gives us all of our possible values of $A_3$ and $B_3$, that is

$$\mathfrak{c}^{-1}\big([A_1, B_1]^{-1}\big) = \big\{(A_3, B_3) \in \mathrm{SU}(2) \,:\, \big[A_3, B_3\big] = \big[A_1, B_1\big]^{-1}\big\}$$

We know that the fibers of the commutator map of SU(2) are connected and so $\mathfrak{c}^{-1}\big([A_1, B_1]^{-1}\big)$ is connected. Next, by our relation involving $A_2$ and $B_2$ we know that the two must commute and elements of SU(2) commute if and only if they lie in the same maximal torus, which in SU(2) is conjugate to the subgroup of diagonal matrices. Let $\mathrm{M}_T$ denote a maximal torus in SU(2),

$$\mathrm{M}_T = \left\{\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta}\end{pmatrix} : \theta \in [0, 2\pi)\right\}$$

Since $A_2, B_2$ commute, there exists $g \in \mathrm{SU}(2)$ such that

$$g\,A_2 g^{-1},\ g\,B_2 g^{-1} \in \mathrm{M}_T$$

This follows from the fact that all maximal tori in SU(2) are conjugate and every element is contained in some maximal torus. Recall that $\mathrm{M}_T$ is connected in SU(2). Thus,

$$\mathcal{F}\big((\Phi^n)^*\big)_\pm \cong \mathrm{SU}(2)^2 \times \mathrm{M}_T^2 \times \mathfrak{c}^{-1}\big([A_1, B_1]^{-1}\big)$$

which is connected as the finite product of connected spaces is connected. □

We have now classified the connectedness of all three of our fixed point stratum sets. Thus our last step in order to fully classify our fixed point set for the character variety is to determine how many distinct connected components are left after taking the union of all such fixed point stratum sets.

**Theorem 2.** *The fixed point set of the n-th power of* $\Phi^*$, *has*

$$\left\lfloor \frac{n^2}{2} \right\rfloor + 1$$

*connected components.*

*Proof.* Fix $n \in \mathbb{Z}\setminus\{0\}$. We begin recalling that if the intersection of two connected sets is nonempty then their union is connected. With this fact consider $\mathcal{F}\big((\Phi^n)^*\big)_{\pm}$ and $\mathcal{F}\big((\Phi^n)^*\big)_{\not\equiv}$. We have shown the each respective set is connected for every $n \in \mathbb{Z} \setminus \{0\}$ and so we will examine their intersection. Observe that if we take each argument of our 6-tuples in our respective sets to be the same element in the center of SU(2) then our relations hold. Therefore we have

$$\big(I, I, I, I, I, I\big),\ \big(-I, -I, -I, -I, -I, -I\big) \in \mathcal{F}\big((\Phi^n)^*\big)_{\pm} \cap \mathcal{F}\big((\Phi^n)^*\big)_{\not\equiv}$$

and so our intersection is nonempty. Hence

$$\mathcal{F}\big((\Phi^n)^*\big)_{\pm} \cup \mathcal{F}\big((\Phi^n)^*\big)_{\not\equiv}$$

is connected. Next let us consider our fixed point set of the representation variety, specifically, the end cap of our large connected component which corresponds to that which we pieced together in **Theorem 1.** Note that this end cap, the fixed point stratum set

$$\{I\} \times \mathcal{F}_I^+ = \big\{A_1'\big\} \times \Big\{(B_1, A_2, B_2, A_3, B_3) \in \mathrm{SU}(2)^5 \ :\ \big[A_2, B_2\big] = \big[A_3, B_3\big]^{-1},\ \big([A_3, B_3]\big)^n = I\Big\}$$

contains the 6-tuple $(I, I, I, I, I, I)$. Therefore, we can connect our other two stratum of the fixed point set of the character variety

$$\mathcal{F}\big((\Phi^n)^*\big)_{\pm} \quad \text{and} \quad \mathcal{F}\big((\Phi^n)^*\big)_{\not\equiv}$$

to this set. Thus we get one connected component

$$\{I\} \times \mathcal{F}_I^+ \cup \mathcal{F}\big((\Phi^n)^*\big)_{\pm} \cup \mathcal{F}\big((\Phi^n)^*\big)_{\not\equiv}$$

However, as we previously noted, $\{I\} \times \mathcal{F}_I^+$ is just one of the two end caps of the larger connected component in the fixed point set of the representation variety, thus

$$\mathcal{F}\big((\Phi^n)^*\big)_{\pm} \quad \text{and} \quad \mathcal{F}\big((\Phi^n)^*\big)_{\not\equiv}$$

just get added to this component. Hence, we are left with the

$$\left\lfloor \frac{|n|^2}{2} \right\rfloor + 1$$

connected components of our fixed point set of the representation variety and so we are done. □

## 6. Dehn Twists about Simply Intersecting Curves

**6.**

- $T_{\gamma_1}:$
  $\alpha_1 \mapsto \beta_1^{-1}\beta_3^{-1}\alpha_1$ $\qquad \beta_1 \mapsto \beta_1$
  $\alpha_2 \mapsto \beta_3^{-1}\beta_1^{-1}\alpha_2\beta_1\beta_3$ $\qquad \beta_2 \mapsto \beta_3^{-1}\beta_1^{-1}\beta_2\beta_1\beta_3$
  $\alpha_3 \mapsto \beta_3^{-1}\beta_1^{-1}\alpha_3$ $\qquad \beta_3 \mapsto \beta_3$
- $T_{\gamma_2}:$
  $\alpha_1 \mapsto \beta_1^{-1}\beta_2^{-1}\alpha_1$ $\qquad \beta_1 \mapsto \beta_1$
  $\alpha_2 \mapsto \beta_2^{-1}\beta_1^{-1}\alpha_2$ $\qquad \beta_2 \mapsto \beta_2$
  $\alpha_3 \mapsto \beta_1^{-1}\beta_2^{-1}\alpha_3\beta_2\beta_1$ $\qquad \beta_3 \mapsto \beta_1^{-1}\beta_2^{-1}\beta_3\beta_2\beta_1$

- $T_{\gamma_1}^{-1}$ : $\alpha_1 \mapsto \beta_3\beta_1\alpha_1$ $\beta_1 \mapsto \beta_1$
  $\alpha_2 \mapsto \beta_1\beta_3\alpha_2\beta_3^{-1}\beta_1^{-1}$ $\beta_2 \mapsto \beta_1\beta_3\beta_2\beta_3^{-1}\beta_1^{-1}$
  $\alpha_3 \mapsto \beta_1\beta_3\alpha_3$ $\beta_3 \mapsto \beta_3$

- $T_{\gamma_2}^{-1}$ : $\alpha_1 \mapsto \beta_2\beta_1\alpha_1$ $\beta_1 \mapsto \beta_1$
  $\alpha_2 \mapsto \beta_1\beta_2\alpha_2$ $\beta_2 \mapsto \beta_2$
  $\alpha_3 \mapsto \beta_2\beta_1\alpha_3\beta_1^{-1}\beta_2^{-1}$ $\beta_3 \mapsto \beta_2\beta_1\beta_3\beta_1^{-1}\beta_2^{-1}$

- $T_{\gamma_1} \circ T_{\gamma_2} \circ T_{\gamma_1}^{-1} \circ T_{\gamma_2}^{-1}$ :

$$\begin{aligned}
&\alpha_1 \mapsto \beta_3^{-1}\beta_1^{-1}\beta_2^{-1}\beta_1\beta_3\beta_1^{-1}\beta_2\beta_1\alpha_1\\
&\beta_1 \mapsto \beta_1\\
&\alpha_2 \mapsto [\beta_1\beta_3^{-1}\beta_1^{-1}, \beta_2^{-1}]\alpha_2[\beta_3^{-1}\beta_1^{-1}\beta_2^{-1}, \beta_1\beta_3\beta_1^{-1}]^{-1}\\
&\beta_2 \mapsto [\beta_3^{-1}\beta_1^{-1}\beta_2^{-1}, \beta_1\beta_3\beta_1^{-1}]\beta_2[\beta_3^{-1}\beta_1^{-1}\beta_2^{-1}, \beta_1\beta_3\beta_1^{-1}]^{-1}\\
&\alpha_3 \mapsto [\beta_3^{-1}\beta_1^{-1}\beta_2^{-1}, \beta_1\beta_3\beta_1^{-1}]\beta_2[\beta_3^{-1}\beta_1^{-1}\beta_2^{-1}, \beta_1\beta_3\beta_1^{-1}]^{-1}\beta_1\beta_3^{-1}\beta_1^{-1}\beta_2^{-1}\beta_1\beta_3\beta_1^{-1}\alpha_3\\
&\qquad [\beta_2\beta_1\beta_3, \beta_1^{-1}]^{-1}[\beta_3^{-1}\beta_1^{-1}\beta_2^{-1}, \beta_1\beta_3\beta_1^{-1}]^{-1}\\
&\beta_3 \mapsto [\beta_3^{-1}\beta_1^{-1}\beta_2^{-1}, \beta_1\beta_3\beta_1^{-1}][\beta_2\beta_1\beta_3, \beta_1^{-1}]\beta_3[\beta_2\beta_1\beta_3, \beta_1^{-1}]^{-1}[\beta_3^{-1}\beta_1^{-1}\beta_2^{-1}, \beta_1\beta_3\beta_1^{-1}]^{-1}
\end{aligned}$$

### 6.1. Fixed Point Equations for $\left(T_{\gamma_1} \circ T_{\gamma_2} \circ T_{\gamma_1}^{-1} \circ T_{\gamma_2}^{-1}\right)^*$:



- On $R(\Sigma, G)$ :

$$\begin{aligned}
&A_1 \mapsto B_3^{-1}B_1^{-1}B_2^{-1}B_1B_3B_1^{-1}B_2B_1A_1\\
&B_1 \mapsto B_1\\
&A_2 \mapsto [B_1B_3^{-1}B_1^{-1}, B_2^{-1}]A_2[B_3^{-1}B_1^{-1}B_2^{-1}, B_1B_3B_1^{-1}]^{-1}\\
&B_2 \mapsto [B_3^{-1}B_1^{-1}B_2^{-1}, B_1B_3B_1^{-1}]B_2[B_3^{-1}B_1^{-1}B_2^{-1}, B_1B_3B_1^{-1}]^{-1}\\
&A_3 \mapsto [B_3^{-1}B_1^{-1}B_2^{-1}, B_1B_3B_1^{-1}]B_2[B_3^{-1}B_1^{-1}B_2^{-1}, B_1B_3B_1^{-1}]^{-1}B_1B_3^{-1}B_1^{-1}B_2^{-1}B_1B_3B_1^{-1}A_3\\
&\qquad [B_2B_1B_3, B_1^{-1}]^{-1}[B_3^{-1}B_1^{-1}B_2^{-1}, B_1B_3B_1^{-1}]^{-1}\\
&B_3 \mapsto [B_3^{-1}B_1^{-1}B_2^{-1}, B_1B_3B_1^{-1}][B_2B_1B_3, B_1^{-1}]B_3[B_2B_1B_3, B_1^{-1}]^{-1}[B_3^{-1}B_1^{-1}B_2^{-1}, B_1B_3B_1^{-1}]^{-1}
\end{aligned}$$

- On $R(\Sigma, G)/G$ :

$$\begin{aligned}
&T^{-1}A_1T \mapsto B_3^{-1}B_1^{-1}B_2^{-1}B_1B_3B_1^{-1}B_2B_1A_1\\
&T^{-1}B_1T \mapsto B_1\\
&T^{-1}A_2T \mapsto [B_1B_3^{-1}B_1^{-1}, B_2^{-1}]A_2[B_3^{-1}B_1^{-1}B_2^{-1}, B_1B_3B_1^{-1}]^{-1}\\
&T^{-1}B_2T \mapsto [B_3^{-1}B_1^{-1}B_2^{-1}, B_1B_3B_1^{-1}]B_2[B_3^{-1}B_1^{-1}B_2^{-1}, B_1B_3B_1^{-1}]^{-1}\\
&T^{-1}A_3T \mapsto [B_3^{-1}B_1^{-1}B_2^{-1}, B_1B_3B_1^{-1}]B_2[B_3^{-1}B_1^{-1}B_2^{-1}, B_1B_3B_1^{-1}]^{-1}B_1B_3^{-1}B_1^{-1}B_2^{-1}B_1B_3B_1^{-1}\\
&\qquad A_3[B_2B_1B_3, B_1^{-1}]^{-1}[B_3^{-1}B_1^{-1}B_2^{-1}, B_1B_3B_1^{-1}]^{-1}\\
&T^{-1}B_3T \mapsto [B_3^{-1}B_1^{-1}B_2^{-1}, B_1B_3B_1^{-1}][B_2B_1B_3, B_1^{-1}]B_3[B_2B_1B_3, B_1^{-1}]^{-1}\\
&\qquad [B_3^{-1}B_1^{-1}B_2^{-1}, B_1B_3B_1^{-1}]^{-1}
\end{aligned}$$

### 6.2. Computing $\Phi = T_{\gamma_1} \circ T_{\gamma_2} \circ T_{\gamma_1}^{-1} \circ T_{\gamma_2}^{-1}$.

- $\Phi(\alpha_1)$ :

$$\begin{aligned}
T_{\gamma_2}^{-1}(\alpha_1) &= \beta_2\beta_1\alpha_1 \\
T_{\gamma_1}^{-1}\big(T_{\gamma_2}^{-1}(\alpha_1)\big) &= T_{\gamma_1}^{-1}(\beta_2\beta_1\alpha_1) \\
&= T_{\gamma_1}^{-1}(\beta_2)T_{\gamma_1}^{-1}(\beta_1)T_{\gamma_1}^{-1}(\alpha_1) \\
&= \beta_1\beta_3\beta_2\beta_3^{-1}\beta_1^{-1}\beta_1\beta_3\beta_1\alpha_1 \\
&= \beta_1\beta_3\beta_2\beta_1\alpha_1 \\
T_{\gamma_2}\big(T_{\gamma_1}^{-1}\big(T_{\gamma_2}^{-1}(\alpha_1)\big)\big) &= T_{\gamma_2}(\beta_1\beta_3\beta_2\beta_1\alpha_1) \\
&= T_{\gamma_2}(\beta_1)T_{\gamma_2}(\beta_3)T_{\gamma_2}(\beta_2)T_{\gamma_2}(\beta_1)T_{\gamma_2}(\alpha_1) \\
&= \beta_1\beta_1^{-1}\beta_2^{-1}\beta_3\beta_2\beta_1\beta_2\beta_1\beta_1^{-1}\beta_2^{-1}\alpha_1 \\
&= \beta_2^{-1}\beta_3\beta_2\beta_1\alpha_1 \\
T_{\gamma_1}\big(T_{\gamma_2}\big(T_{\gamma_1}^{-1}\big(T_{\gamma_2}^{-1}(\alpha_1)\big)\big)\big) &= T_{\gamma_1}(\beta_2^{-1}\beta_3\beta_2\beta_1\alpha_1) \\
&= T_{\gamma_1}(\beta_2^{-1})T_{\gamma_1}(\beta_3)T_{\gamma_1}(\beta_2)T_{\gamma_1}(\beta_1)T_{\gamma_1}(\alpha_1) \\
&= \big(T_{\gamma_1}(\beta_2)\big)^{-1}T_{\gamma_1}(\beta_3)T_{\gamma_1}(\beta_2)T_{\gamma_1}(\beta_1)T_{\gamma_1}(\alpha_1) \\
&= \beta_3^{-1}\beta_1^{-1}\beta_2^{-1}\beta_1\beta_3\beta_3\beta_3^{-1}\beta_1^{-1}\beta_2\beta_1\beta_3\beta_1\beta_1^{-1}\beta_3^{-1}\alpha_1 \\
&= \beta_3^{-1}\beta_1^{-1}\beta_2^{-1}\beta_1\beta_3\beta_1^{-1}\beta_2\beta_1\alpha_1
\end{aligned}$$

$$\boxed{\Phi(\alpha_1) = \beta_3^{-1}\beta_1^{-1}\beta_2^{-1}\beta_1\beta_3\beta_1^{-1}\beta_2\beta_1\alpha_1}$$

- $\Phi(\beta_1)$ :

$$\begin{aligned}
T_{\gamma_2}^{-1}(\beta_1) &= \beta_1 \\
T_{\gamma_1}^{-1}\big(T_{\gamma_2}^{-1}(\beta_1)\big) &= T_{\gamma_1}^{-1}(\beta_1) \\
&= \beta_1 \\
T_{\gamma_2}\big(T_{\gamma_1}^{-1}\big(T_{\gamma_2}^{-1}(\beta_1)\big)\big) &= T_{\gamma_2}(\beta_1) \\
&= \beta_1 \\
T_{\gamma_1}\big(T_{\gamma_2}\big(T_{\gamma_1}^{-1}\big(T_{\gamma_2}^{-1}(\beta_1)\big)\big)\big) &= T_{\gamma_1}(\beta_1) \\
&= \beta_1
\end{aligned}$$

$$\boxed{\Phi(\beta_1) = \beta_1}$$

- $\Phi(\alpha_2)$ :

$$
\begin{aligned}
T_{\gamma_2}^{-1}(\alpha_2) &= \beta_1\beta_2\alpha_2 \\
T_{\gamma_1}^{-1}\big(T_{\gamma_2}^{-1}(\alpha_2)\big) &= T_{\gamma_1}^{-1}(\beta_1\beta_2\alpha_2) \\
&= T_{\gamma_1}^{-1}(\beta_1)T_{\gamma_1}^{-1}(\beta_2)T_{\gamma_1}^{-1}(\alpha_2) \\
&= \beta_1\beta_1\beta_3\beta_2\beta_3^{-1}\beta_1^{-1}\beta_1\beta_3\alpha_2\beta_3^{-1}\beta_1^{-1} \\
&= \beta_1\beta_1\beta_3\beta_2\alpha_2\beta_3^{-1}\beta_1^{-1} \\
T_{\gamma_2}\big(T_{\gamma_1}^{-1}\big(T_{\gamma_2}^{-1}(\alpha_2)\big)\big) &= T_{\gamma_2}(\beta_1\beta_1\beta_3\beta_2\alpha_2\beta_3^{-1}\beta_1^{-1}) \\
&= T_{\gamma_2}(\beta_1)T_{\gamma_2}(\beta_1)T_{\gamma_2}(\beta_3)T_{\gamma_2}(\beta_2)T_{\gamma_2}(\alpha_2)T_{\gamma_2}(\beta_3^{-1})T_{\gamma_2}(\beta_1^{-1}) \\
&= T_{\gamma_2}(\beta_1)T_{\gamma_2}(\beta_1)T_{\gamma_2}(\beta_3)T_{\gamma_2}(\beta_2)T_{\gamma_2}(\alpha_2)\big(T_{\gamma_2}(\beta_3)\big)^{-1}\big(T_{\gamma_2}(\beta_1)\big)^{-1} \\
&= \beta_1\beta_1\beta_1^{-1}\beta_2^{-1}\beta_3\beta_2\beta_1\beta_2\beta_2^{-1}\beta_1^{-1}\alpha_2\beta_1^{-1}\beta_2^{-1}\beta_3^{-1}\beta_2\beta_1\beta_1^{-1} \\
&= \beta_1\beta_2^{-1}\beta_3\beta_2\alpha_2\beta_1^{-1}\beta_2^{-1}\beta_3^{-1}\beta_2 \\
T_{\gamma_1}\big(T_{\gamma_2}\big(T_{\gamma_1}^{-1}\big(T_{\gamma_2}^{-1}(\alpha_2)\big)\big)\big) &= T_{\gamma_1}(\beta_1\beta_2^{-1}\beta_3\beta_2\alpha_2\beta_1^{-1}\beta_2^{-1}\beta_3^{-1}\beta_2) \\
&= T_{\gamma_1}(\beta_1)T_{\gamma_1}(\beta_2^{-1})T_{\gamma_1}(\beta_3)T_{\gamma_1}(\beta_2)T_{\gamma_1}(\alpha_2)T_{\gamma_1}(\beta_1^{-1})T_{\gamma_1}(\beta_2^{-1})T_{\gamma_1}(\beta_3^{-1})T_{\gamma_1}(\beta_2) \\
&= T_{\gamma_1}(\beta_1)\big(T_{\gamma_1}(\beta_2)\big)^{-1}T_{\gamma_1}(\beta_3)T_{\gamma_1}(\beta_2)T_{\gamma_1}(\alpha_2)\big(T_{\gamma_1}(\beta_1)\big)^{-1}\big(T_{\gamma_1}(\beta_2)\big)^{-1}\big(T_{\gamma_1}(\beta_3)\big)^{-1}T_{\gamma_1}(\beta_2) \\
&= \beta_1\beta_3^{-1}\beta_1^{-1}\beta_2^{-1}\beta_1\beta_3\beta_3\beta_3^{-1}\beta_1^{-1}\beta_2\beta_1\beta_3\beta_3^{-1}\beta_1^{-1}\alpha_2\beta_1\beta_3\beta_1^{-1}\beta_3^{-1}\beta_1^{-1}\beta_2^{-1}\beta_1\beta_3\beta_3^{-1}\beta_3^{-1}\beta_1^{-1}\beta_2\beta_1\beta_3 \\
&= \beta_1\beta_3^{-1}\beta_1^{-1}\beta_2^{-1}\beta_1\beta_3\beta_1^{-1}\beta_2\alpha_2\beta_1\beta_3\beta_1^{-1}\beta_3^{-1}\beta_1^{-1}\beta_2^{-1}\beta_1\beta_3^{-1}\beta_1^{-1}\beta_2\beta_1\beta_3 \\
&= [\beta_1\beta_3^{-1}\beta_1^{-1},\beta_2^{-1}]\alpha_2[\beta_1\beta_3\beta_1^{-1},\beta_3^{-1}\beta_1^{-1}\beta_2^{-1}] \\
&= [\beta_1\beta_3^{-1}\beta_1^{-1},\beta_2^{-1}]\alpha_2[\beta_3^{-1}\beta_1^{-1}\beta_2^{-1},\beta_1\beta_3\beta_1^{-1}]^{-1}
\end{aligned}
$$

$$
\boxed{\Phi(\alpha_2) = [\beta_1\beta_3^{-1}\beta_1^{-1},\beta_2^{-1}]\alpha_2[\beta_3^{-1}\beta_1^{-1}\beta_2^{-1},\beta_1\beta_3\beta_1^{-1}]^{-1}}
$$

- $\Phi(\beta_2)$ :

$$
\begin{aligned}
T_{\gamma_2}^{-1}(\beta_2) &= \beta_2 \\
T_{\gamma_1}^{-1}\big(T_{\gamma_2}^{-1}(\beta_2)\big) &= T_{\gamma_1}^{-1}(\beta_2) \\
&= \beta_1\beta_3\beta_2\beta_3^{-1}\beta_1^{-1} \\
T_{\gamma_2}\big(T_{\gamma_1}^{-1}\big(T_{\gamma_2}^{-1}(\beta_2)\big)\big) &= T_{\gamma_2}(\beta_1\beta_3\beta_2\beta_3^{-1}\beta_1^{-1}) \\
&= T_{\gamma_2}(\beta_1)T_{\gamma_2}(\beta_3)T_{\gamma_2}(\beta_2)T_{\gamma_2}(\beta_3^{-1})T_{\gamma_2}(\beta_1^{-1}) \\
&= T_{\gamma_2}(\beta_1)T_{\gamma_2}(\beta_3)T_{\gamma_2}(\beta_2)\big(T_{\gamma_2}(\beta_3)\big)^{-1}\big(T_{\gamma_2}(\beta_1)\big)^{-1} \\
&= \beta_1\beta_1^{-1}\beta_2^{-1}\beta_3\beta_2\beta_1\beta_2\beta_1^{-1}\beta_2^{-1}\beta_3^{-1}\beta_2\beta_1\beta_1^{-1} \\
&= \beta_2^{-1}\beta_3\beta_2\beta_1\beta_2\beta_1^{-1}\beta_2^{-1}\beta_3^{-1}\beta_2 \\
T_{\gamma_1}\big(T_{\gamma_2}\big(T_{\gamma_1}^{-1}\big(T_{\gamma_2}^{-1}(\beta_2)\big)\big)\big) &= T_{\gamma_1}(\beta_2^{-1}\beta_3\beta_2\beta_1\beta_2\beta_1^{-1}\beta_2^{-1}\beta_3^{-1}\beta_2) \\
&= T_{\gamma_1}(\beta_2^{-1})T_{\gamma_1}(\beta_3)T_{\gamma_1}(\beta_2)T_{\gamma_1}(\beta_1)T_{\gamma_1}(\beta_2)T_{\gamma_1}(\beta_1^{-1})T_{\gamma_1}(\beta_2^{-1})T_{\gamma_1}(\beta_3^{-1})T_{\gamma_1}(\beta_2) \\
&= \big(T_{\gamma_1}(\beta_2)\big)^{-1}T_{\gamma_1}(\beta_3)T_{\gamma_1}(\beta_2)T_{\gamma_1}(\beta_1)T_{\gamma_1}(\beta_2)\big(T_{\gamma_1}(\beta_1)\big)^{-1}\big(T_{\gamma_1}(\beta_2)\big)^{-1}\big(T_{\gamma_1}(\beta_3)\big)^{-1}T_{\gamma_1}(\beta_2) \\
&= \beta_3^{-1}\beta_1^{-1}\beta_2^{-1}\beta_1\beta_3\beta_3\beta_3^{-1}\beta_1^{-1}\beta_2\beta_1\beta_3\beta_1\beta_3^{-1}\beta_1^{-1}\beta_2\beta_1\beta_3\beta_1^{-1}\beta_3^{-1}\beta_1^{-1}\beta_2^{-1}\beta_1\beta_3\beta_3^{-1}\beta_3^{-1}\beta_1^{-1}\beta_2\beta_1\beta_3 \\
&= \beta_3^{-1}\beta_1^{-1}\beta_2^{-1}\beta_1\beta_3\beta_1^{-1}\beta_2\beta_1\beta_3\beta_1\beta_3^{-1}\beta_1^{-1}\beta_2\beta_1\beta_3\beta_1^{-1}\beta_3^{-1}\beta_1^{-1}\beta_2^{-1}\beta_1\beta_3^{-1}\beta_1^{-1}\beta_2\beta_1\beta_3 \\
&= [\beta_3^{-1}\beta_1^{-1}\beta_2^{-1},\beta_1\beta_3\beta_1^{-1}]\beta_2[\beta_1\beta_3\beta_1^{-1},\beta_3^{-1}\beta_1^{-1}\beta_2^{-1}] \\
&= [\beta_3^{-1}\beta_1^{-1}\beta_2^{-1},\beta_1\beta_3\beta_1^{-1}]\beta_2[\beta_3^{-1}\beta_1^{-1}\beta_2^{-1},\beta_1\beta_3\beta_1^{-1}]^{-1}
\end{aligned}
$$

$$\boxed{\Phi(\beta_2) = [\beta_3^{-1}\beta_1^{-1}\beta_2^{-1}, \beta_1\beta_3\beta_1^{-1}]\beta_2[\beta_3^{-1}\beta_1^{-1}\beta_2^{-1}, \beta_1\beta_3\beta_1^{-1}]^{-1}}$$

- $\Phi(\alpha_3)$ :

$$\begin{aligned}
T_{\gamma_2}^{-1}(\alpha_3) &= \beta_2\beta_1\alpha_3\beta_1^{-1}\beta_2^{-1} \\
T_{\gamma_1}^{-1}\big(T_{\gamma_2}^{-1}(\alpha_3)\big) &= T_{\gamma_1}^{-1}(\beta_2\beta_1\alpha_3\beta_1^{-1}\beta_2^{-1}) \\
&= T_{\gamma_1}^{-1}(\beta_2)T_{\gamma_1}^{-1}(\beta_1)T_{\gamma_1}^{-1}(\alpha_3)T_{\gamma_1}^{-1}(\beta_1^{-1})T_{\gamma_1}^{-1}(\beta_2^{-1}) \\
&= T_{\gamma_1}^{-1}(\beta_2)T_{\gamma_1}^{-1}(\beta_1)T_{\gamma_1}^{-1}(\alpha_3)\big(T_{\gamma_1}^{-1}(\beta_1)\big)^{-1}\big(T_{\gamma_1}^{-1}(\beta_2)\big)^{-1} \\
&= \beta_1\beta_3\beta_2\beta_3^{-1}\beta_1^{-1}\beta_1\beta_1\beta_3\alpha_3\beta_1^{-1}\beta_1\beta_3\beta_2^{-1}\beta_3^{-1}\beta_1^{-1} \\
&= \beta_1\beta_3\beta_2\beta_3^{-1}\beta_1\beta_3\alpha_3\beta_3\beta_2^{-1}\beta_3^{-1}\beta_1^{-1} \\
T_{\gamma_2}\big(T_{\gamma_1}^{-1}\big(T_{\gamma_2}^{-1}(\alpha_3)\big)\big) &= T_{\gamma_2}(\beta_1\beta_3\beta_2\beta_3^{-1}\beta_1\beta_3\alpha_3\beta_3\beta_2^{-1}\beta_3^{-1}\beta_1^{-1}) \\
&= T_{\gamma_2}(\beta_1)T_{\gamma_2}(\beta_3)T_{\gamma_2}(\beta_2)T_{\gamma_2}(\beta_3^{-1})T_{\gamma_2}(\beta_1)T_{\gamma_2}(\beta_3)T_{\gamma_2}(\alpha_3)T_{\gamma_2}(\beta_3)T_{\gamma_2}(\beta_2^{-1})T_{\gamma_2}(\beta_3^{-1})T_{\gamma_2}(\beta_1^{-1}) \\
&= T_{\gamma_2}(\beta_1)T_{\gamma_2}(\beta_3)T_{\gamma_2}(\beta_2)\big(T_{\gamma_2}(\beta_3)\big)^{-1}T_{\gamma_2}(\beta_1)T_{\gamma_2}(\beta_3)T_{\gamma_2}(\alpha_3)T_{\gamma_2}(\beta_3)\big(T_{\gamma_2}(\beta_2)\big)^{-1}\big(T_{\gamma_2}(\beta_3)\big)^{-1} \\
&\quad \big(T_{\gamma_2}(\beta_1)\big)^{-1} \\
&= \beta_1\beta_1^{-1}\beta_2^{-1}\beta_3\beta_2\beta_1\beta_2\beta_1^{-1}\beta_2^{-1}\beta_3^{-1}\beta_2\beta_1\beta_1\beta_1^{-1}\beta_2^{-1}\beta_3\beta_2\beta_1\beta_1^{-1}\beta_2^{-1}\alpha_3\beta_2\beta_1\beta_1^{-1}\beta_2^{-1}\beta_3\beta_2\beta_1\beta_2^{-1}\beta_1^{-1} \\
&\quad \beta_2^{-1}\beta_3^{-1}\beta_2\beta_1\beta_1^{-1} \\
&= \beta_2^{-1}\beta_3\beta_2\beta_1\beta_2\beta_1^{-1}\beta_2^{-1}\beta_3^{-1}\beta_2\beta_1\beta_2^{-1}\beta_3\alpha_3\beta_3\beta_2\beta_1\beta_2^{-1}\beta_1^{-1}\beta_2^{-1}\beta_3^{-1}\beta_2 \\
T_{\gamma_1}\big(T_{\gamma_2}\big(T_{\gamma_1}^{-1}\big(T_{\gamma_2}^{-1}(\alpha_3)\big)\big)\big) &= T_{\gamma_1}(\beta_2^{-1}\beta_3\beta_2\beta_1\beta_2\beta_1^{-1}\beta_2^{-1}\beta_3^{-1}\beta_2\beta_1\beta_2^{-1}\beta_3\alpha_3\beta_3\beta_2\beta_1\beta_2^{-1}\beta_1^{-1}\beta_2^{-1}\beta_3^{-1}\beta_2) \\
&= T_{\gamma_1}(\beta_2^{-1})T_{\gamma_1}(\beta_3)T_{\gamma_1}(\beta_2)T_{\gamma_1}(\beta_1)T_{\gamma_1}(\beta_2)T_{\gamma_1}(\beta_1^{-1})T_{\gamma_1}(\beta_2^{-1})T_{\gamma_1}(\beta_3^{-1})T_{\gamma_1}(\beta_2)T_{\gamma_1}(\beta_1)T_{\gamma_1}(\beta_2^{-1}) \\
&\quad T_{\gamma_1}(\beta_3)T_{\gamma_1}(\alpha_3)T_{\gamma_1}(\beta_3)T_{\gamma_1}(\beta_2)T_{\gamma_1}(\beta_1)T_{\gamma_1}(\beta_2^{-1})T_{\gamma_1}(\beta_1^{-1})T_{\gamma_1}(\beta_2^{-1})T_{\gamma_1}(\beta_3^{-1})T_{\gamma_1}(\beta_2) \\
&= \big(T_{\gamma_1}(\beta_2)\big)^{-1}T_{\gamma_1}(\beta_3)T_{\gamma_1}(\beta_2)T_{\gamma_1}(\beta_1)T_{\gamma_1}(\beta_2)\big(T_{\gamma_1}(\beta_1)\big)^{-1}\big(T_{\gamma_1}(\beta_2)\big)^{-1}\big(T_{\gamma_1}(\beta_3)\big)^{-1}T_{\gamma_1}(\beta_2) \\
&\quad T_{\gamma_1}(\beta_1)\big(T_{\gamma_1}(\beta_2)\big)^{-1}T_{\gamma_1}(\beta_3)T_{\gamma_1}(\alpha_3)T_{\gamma_1}(\beta_3)T_{\gamma_1}(\beta_2)T_{\gamma_1}(\beta_1)\big(T_{\gamma_1}(\beta_2)\big)^{-1}\big(T_{\gamma_1}(\beta_1)\big)^{-1} \\
&\quad \big(T_{\gamma_1}(\beta_2)\big)^{-1}\big(T_{\gamma_1}(\beta_3)\big)^{-1}T_{\gamma_1}(\beta_2) \\
&= \beta_3^{-1}\beta_1^{-1}\beta_2^{-1}\beta_1\beta_3\beta_3\beta_3^{-1}\beta_1^{-1}\beta_2\beta_1\beta_3\beta_1\beta_3^{-1}\beta_1^{-1}\beta_2\beta_1\beta_3\beta_1^{-1}\beta_3^{-1}\beta_1^{-1}\beta_2^{-1}\beta_1\beta_3\beta_3^{-1}\beta_3^{-1}\beta_1^{-1}\beta_2\beta_1\beta_3 \\
&\quad \beta_1\beta_3^{-1}\beta_1^{-1}\beta_2^{-1}\beta_1\beta_3\beta_3\beta_3^{-1}\beta_1^{-1}\alpha_3\beta_3\beta_3^{-1}\beta_1^{-1}\beta_2\beta_1\beta_3\beta_1\beta_3^{-1}\beta_1^{-1}\beta_2^{-1}\beta_1\beta_3\beta_1^{-1}\beta_3^{-1}\beta_1^{-1}\beta_2^{-1}\beta_1\beta_3\beta_3^{-1} \\
&\quad \beta_3^{-1}\beta_1^{-1}\beta_2\beta_1\beta_3 \\
&= \beta_3^{-1}\beta_1^{-1}\beta_2^{-1}\beta_1\beta_3\beta_1^{-1}\beta_2\beta_1\beta_3\beta_1\beta_3^{-1}\beta_1^{-1}\beta_2\beta_1\beta_3\beta_1^{-1}\beta_3^{-1}\beta_1^{-1}\beta_2^{-1}\beta_1\beta_3^{-1}\beta_1^{-1}\beta_2\beta_1\beta_3\beta_1\beta_3^{-1}\beta_1^{-1}\beta_2^{-1} \\
&\quad \beta_1\beta_3\beta_1^{-1}\alpha_3\beta_1^{-1}\beta_2\beta_1\beta_3\beta_1\beta_3^{-1}\beta_1^{-1}\beta_2^{-1}\beta_1\beta_3\beta_1^{-1}\beta_3^{-1}\beta_1^{-1}\beta_2^{-1}\beta_1\beta_3^{-1}\beta_1^{-1}\beta_2\beta_1\beta_3 \\
&= [\beta_3^{-1}\beta_1^{-1}\beta_2^{-1}, \beta_1\beta_3\beta_1^{-1}]\beta_2[\beta_1\beta_3\beta_1^{-1}, \beta_3^{-1}\beta_1^{-1}\beta_2^{-1}]\beta_1\beta_3^{-1}\beta_1^{-1}\beta_2^{-1}\beta_1\beta_3\beta_1^{-1}\alpha_3 \\
&\quad [\beta_1^{-1}, \beta_2\beta_1\beta_3][\beta_1\beta_3\beta_1^{-1}, \beta_3^{-1}\beta_1^{-1}\beta_2^{-1}] \\
&= [\beta_3^{-1}\beta_1^{-1}\beta_2^{-1}, \beta_1\beta_3\beta_1^{-1}]\beta_2[\beta_3^{-1}\beta_1^{-1}\beta_2^{-1}, \beta_1\beta_3\beta_1^{-1}]^{-1}\beta_1\beta_3^{-1}\beta_1^{-1}\beta_2^{-1}\beta_1\beta_3\beta_1^{-1}\alpha_3 \\
&\quad [\beta_2\beta_1\beta_3, \beta_1^{-1}]^{-1}[\beta_3^{-1}\beta_1^{-1}\beta_2^{-1}, \beta_1\beta_3\beta_1^{-1}]^{-1}
\end{aligned}$$

$$\boxed{\begin{aligned}\Phi(\alpha_3) = {} & [\beta_3^{-1}\beta_1^{-1}\beta_2^{-1}, \beta_1\beta_3\beta_1^{-1}]\beta_2[\beta_3^{-1}\beta_1^{-1}\beta_2^{-1}, \beta_1\beta_3\beta_1^{-1}]^{-1}\beta_1\beta_3^{-1}\beta_1^{-1}\beta_2^{-1}\beta_1\beta_3\beta_1^{-1}\alpha_3 \\ & [\beta_2\beta_1\beta_3, \beta_1^{-1}]^{-1}[\beta_3^{-1}\beta_1^{-1}\beta_2^{-1}, \beta_1\beta_3\beta_1^{-1}]^{-1}\end{aligned}}$$

- $\Phi(\beta_3)$ :

$$
\begin{aligned}
T_{\gamma_2}^{-1}(\beta_3) &= \beta_2\beta_1\beta_3\beta_1^{-1}\beta_2^{-1} \\
T_{\gamma_1}^{-1}\big(T_{\gamma_2}^{-1}(\beta_3)\big) &= T_{\gamma_1}^{-1}(\beta_2\beta_1\beta_3\beta_1^{-1}\beta_2^{-1}) \\
&= T_{\gamma_1}^{-1}(\beta_2)T_{\gamma_1}^{-1}(\beta_1)T_{\gamma_1}^{-1}(\beta_3)T_{\gamma_1}^{-1}(\beta_1^{-1})T_{\gamma_1}^{-1}(\beta_2^{-1}) \\
&= T_{\gamma_1}^{-1}(\beta_2)T_{\gamma_1}^{-1}(\beta_1)T_{\gamma_1}^{-1}(\beta_3)\big(T_{\gamma_1}^{-1}(\beta_1)\big)^{-1}\big(T_{\gamma_1}^{-1}(\beta_2)\big)^{-1} \\
&= \beta_1\beta_3\beta_2\beta_3^{-1}\beta_1^{-1}\beta_1\beta_3\beta_1^{-1}\beta_1\beta_3\beta_2^{-1}\beta_3^{-1}\beta_1^{-1} \\
&= \beta_1\beta_3\beta_2\beta_3\beta_2^{-1}\beta_3^{-1}\beta_1^{-1} \\
T_{\gamma_2}\big(T_{\gamma_1}^{-1}\big(T_{\gamma_2}^{-1}(\beta_3)\big)\big) &= T_{\gamma_2}(\beta_1\beta_3\beta_2\beta_3\beta_2^{-1}\beta_3^{-1}\beta_1^{-1}) \\
&= T_{\gamma_2}(\beta_1)T_{\gamma_2}(\beta_3)T_{\gamma_2}(\beta_2)T_{\gamma_2}(\beta_3)T_{\gamma_2}(\beta_2^{-1})T_{\gamma_2}(\beta_3^{-1})T_{\gamma_2}(\beta_1^{-1}) \\
&= T_{\gamma_2}(\beta_1)T_{\gamma_2}(\beta_3)T_{\gamma_2}(\beta_2)T_{\gamma_2}(\beta_3)\big(T_{\gamma_2}(\beta_2)\big)^{-1}\big(T_{\gamma_2}(\beta_3)\big)^{-1}\big(T_{\gamma_2}(\beta_1)\big)^{-1} \\
&= \beta_1\beta_1^{-1}\beta_2^{-1}\beta_3\beta_2\beta_1\beta_2\beta_1^{-1}\beta_2^{-1}\beta_3\beta_2\beta_1\beta_2^{-1}\beta_1^{-1}\beta_2^{-1}\beta_3^{-1}\beta_2\beta_1\beta_1^{-1} \\
&= \beta_2^{-1}\beta_3\beta_2\beta_1\beta_2\beta_1^{-1}\beta_2^{-1}\beta_3\beta_2\beta_1\beta_2^{-1}\beta_1^{-1}\beta_2^{-1}\beta_3^{-1}\beta_2 \\
T_{\gamma_1}\big(T_{\gamma_2}\big(T_{\gamma_1}^{-1}\big(T_{\gamma_2}^{-1}(\beta_3)\big)\big)\big) &= T_{\gamma_1}(\beta_2^{-1}\beta_3\beta_2\beta_1\beta_2\beta_1^{-1}\beta_2^{-1}\beta_3\beta_2\beta_1\beta_2^{-1}\beta_1^{-1}\beta_2^{-1}\beta_3^{-1}\beta_2) \\
&= T_{\gamma_1}(\beta_2^{-1})T_{\gamma_1}(\beta_3)T_{\gamma_1}(\beta_2)T_{\gamma_1}(\beta_1)T_{\gamma_1}(\beta_2)T_{\gamma_1}(\beta_1^{-1})T_{\gamma_1}(\beta_2^{-1})T_{\gamma_1}(\beta_3)T_{\gamma_1}(\beta_2)T_{\gamma_1}(\beta_1)T_{\gamma_1}(\beta_2^{-1}) \\
&\quad T_{\gamma_1}(\beta_1^{-1})T_{\gamma_1}(\beta_2^{-1})T_{\gamma_1}(\beta_3^{-1})T_{\gamma_1}(\beta_2) \\
&= \big(T_{\gamma_1}(\beta_2)\big)^{-1}T_{\gamma_1}(\beta_3)T_{\gamma_1}(\beta_2)T_{\gamma_1}(\beta_1)T_{\gamma_1}(\beta_2)\big(T_{\gamma_1}(\beta_1)\big)^{-1}\big(T_{\gamma_1}(\beta_2)\big)^{-1}T_{\gamma_1}(\beta_3)T_{\gamma_1}(\beta_2)T_{\gamma_1}(\beta_1) \\
&\quad \big(T_{\gamma_1}(\beta_2)\big)^{-1}\big(T_{\gamma_1}(\beta_1)\big)^{-1}\big(T_{\gamma_1}(\beta_2)\big)^{-1}\big(T_{\gamma_1}(\beta_3)\big)^{-1}T_{\gamma_1}(\beta_2) \\
&= \beta_3^{-1}\beta_1^{-1}\beta_2^{-1}\beta_1\beta_3\beta_3\beta_3^{-1}\beta_1^{-1}\beta_2\beta_1\beta_3\beta_1\beta_3^{-1}\beta_1^{-1}\beta_2\beta_1\beta_3\beta_1^{-1}\beta_3^{-1}\beta_1^{-1}\beta_2^{-1}\beta_1\beta_3\beta_3\beta_3^{-1}\beta_1^{-1}\beta_2\beta_1\beta_3\beta_1 \\
&\quad \beta_3^{-1}\beta_1^{-1}\beta_2^{-1}\beta_1\beta_3\beta_1^{-1}\beta_3^{-1}\beta_1^{-1}\beta_2^{-1}\beta_1\beta_3\beta_3^{-1}\beta_3^{-1}\beta_1^{-1}\beta_2\beta_1\beta_3 \\
&= \beta_3^{-1}\beta_1^{-1}\beta_2^{-1}\beta_1\beta_3\beta_1^{-1}\beta_2\beta_1\beta_3\beta_1\beta_3^{-1}\beta_1^{-1}\beta_2\beta_1\beta_3\beta_1^{-1}\beta_3^{-1}\beta_1^{-1}\beta_2^{-1}\beta_1\beta_3 \\
&\quad \beta_1^{-1}\beta_2\beta_1\beta_3\beta_1\beta_3^{-1}\beta_1^{-1}\beta_2^{-1}\beta_1\beta_3\beta_1^{-1}\beta_3^{-1}\beta_1^{-1}\beta_2^{-1}\beta_1\beta_3^{-1}\beta_1^{-1}\beta_2\beta_1\beta_3 \\
&= [\beta_3^{-1}\beta_1^{-1}\beta_2^{-1}, \beta_1\beta_3\beta_1^{-1}][\beta_2\beta_1\beta_3, \beta_1^{-1}]\beta_3[\beta_1^{-1}, \beta_2\beta_1\beta_3][\beta_3^{-1}\beta_1^{-1}\beta_2^{-1}, \beta_1\beta_3\beta_1^{-1}]^{-1} \\
&= [\beta_3^{-1}\beta_1^{-1}\beta_2^{-1}, \beta_1\beta_3\beta_1^{-1}][\beta_2\beta_1\beta_3, \beta_1^{-1}]\beta_3[\beta_2\beta_1\beta_3, \beta_1^{-1}]^{-1}[\beta_3^{-1}\beta_1^{-1}\beta_2^{-1}, \beta_1\beta_3\beta_1^{-1}]^{-1}
\end{aligned}
$$

$$
\boxed{\Phi(\beta_3) = [\beta_3^{-1}\beta_1^{-1}\beta_2^{-1}, \beta_1\beta_3\beta_1^{-1}][\beta_2\beta_1\beta_3, \beta_1^{-1}]\beta_3[\beta_2\beta_1\beta_3, \beta_1^{-1}]^{-1}[\beta_3^{-1}\beta_1^{-1}\beta_2^{-1}, \beta_1\beta_3\beta_1^{-1}]^{-1}}
$$

6.3. **Checking Relation.**

6.3. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .

- $T_{\gamma_1}$ :

$$
\begin{gathered}
\big[\beta_1^{-1}\beta_3^{-1}\alpha_1,\ \beta_1\big]\beta_3^{-1}\beta_1^{-1}\big[\alpha_2, \beta_2\big]\beta_1\beta_3\big[\beta_3^{-1}\beta_1^{-1}\alpha_3,\ \beta_3\big] \\
\beta_1^{-1}\beta_3^{-1}\alpha_1\beta_1\alpha_1^{-1}\beta_3\beta_1\beta_1^{-1}\beta_3^{-1}\beta_1^{-1}\big[\alpha_2, \beta_2\big]\beta_1\beta_3\beta_3^{-1}\beta_1^{-1}\alpha_3\beta_3\alpha_3^{-1}\beta_1\beta_3\beta_3^{-1} \\
\beta_1^{-1}\beta_3^{-1}\alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1}\big[\alpha_2, \beta_2\big]\alpha_3\beta_3\alpha_3^{-1}\beta_3^{-1}\beta_3\beta_1 \\
\beta_1^{-1}\beta_3^{-1}\big[\alpha_1, \beta_1\big]\big[\alpha_2, \beta_2\big]\alpha_3\beta_3\alpha_3^{-1}\beta_3^{-1}\beta_3\beta_1 \\
\beta_1^{-1}\beta_3^{-1}\big[\alpha_1, \beta_1\big]\big[\alpha_2, \beta_2\big]\big[\alpha_3, \beta_3\big]\beta_3\beta_1 \\
\beta_1^{-1}\beta_3^{-1}\beta_3\beta_1 \\
\boxed{1}
\end{gathered}
$$

- $T_{\gamma_2}$ :

$$\big[\beta_1^{-1}\beta_2^{-1}\alpha_1,\ \beta_1\big]\big[\beta_2^{-1}\beta_1^{-1}\alpha_2,\ \beta_2\big]\beta_1^{-1}\beta_2^{-1}\big[\alpha_3,\beta_3\big]\beta_2\beta_1$$
$$\beta_1^{-1}\beta_2^{-1}\alpha_1\beta_1\alpha_1^{-1}\beta_2\beta_1\beta_1^{-1}\beta_2^{-1}\beta_1^{-1}\alpha_2\beta_2\alpha_2^{-1}\beta_1\beta_2\beta_2^{-1}\beta_1^{-1}\beta_2^{-1}\big[\alpha_3,\beta_3\big]\beta_2\beta_1$$
$$\beta_1^{-1}\beta_2^{-1}\alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1}\alpha_2\beta_2\alpha_2^{-1}\beta_2^{-1}\big[\alpha_3,\beta_3\big]\beta_2\beta_1$$
$$\beta_1^{-1}\beta_2^{-1}\big[\alpha_1,\beta_1\big]\big[\alpha_2,\beta_2\big]\big[\alpha_3,\beta_3\big]\beta_2\beta_1$$
$$\beta_1^{-1}\beta_2^{-1}\beta_2\beta_1$$
$$\boxed{1}$$

- $T_{\gamma_1}^{-1}$ :

$$\big[\beta_3\beta_1\alpha_1,\ \beta_1\big]\beta_1\beta_3\big[\alpha_2,\beta_2\big]\beta_3^{-1}\beta_1^{-1}\big[\beta_1\beta_3\alpha_3,\ \beta_3\big]$$
$$\beta_3\beta_1\alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1}\beta_3^{-1}\beta_1^{-1}\beta_1\beta_3\big[\alpha_2,\beta_2\big]\beta_3^{-1}\beta_1^{-1}\beta_1\beta_3\alpha_3\beta_3\alpha_3^{-1}\beta_3^{-1}\beta_1^{-1}\beta_3^{-1}$$
$$\beta_3\beta_1\alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1}\big[\alpha_2,\beta_2\big]\alpha_3\beta_3\alpha_3^{-1}\beta_3^{-1}\beta_1^{-1}\beta_3^{-1}$$
$$\beta_3\beta_1\big[\alpha_1,\beta_1\big]\big[\alpha_2,\beta_2\big]\big[\alpha_3,\beta_3\big]\beta_1^{-1}\beta_3^{-1}$$
$$\beta_3\beta_1\beta_1^{-1}\beta_3^{-1}$$
$$\boxed{1}$$

- $T_{\gamma_2}^{-1}$ :

$$\big[\beta_2\beta_1\alpha_1,\ \beta_1\big]\big[\beta_1\beta_2\alpha_2,\ \beta_2\big]\beta_2\beta_1\big[\alpha_3,\beta_3\big]\beta_1^{-1}\beta_2^{-1}$$
$$\beta_2\beta_1\alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1}\beta_2^{-1}\beta_1^{-1}\beta_1\beta_2\alpha_2\beta_2\alpha_2^{-1}\beta_2^{-1}\beta_1^{-1}\beta_2^{-1}\beta_2\beta_1\big[\alpha_3,\beta_3\big]\beta_1^{-1}\beta_2^{-1}$$
$$\beta_2\beta_1\alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1}\alpha_2\beta_2\alpha_2^{-1}\beta_2^{-1}\big[\alpha_3,\beta_3\big]\beta_1^{-1}\beta_2^{-1}$$
$$\beta_2\beta_1\big[\alpha_1,\beta_1\big]\big[\alpha_2,\beta_2\big]\big[\alpha_3,\beta_3\big]\beta_1^{-1}\beta_2^{-1}$$
$$\beta_2\beta_1\beta_1^{-1}\beta_2^{-1}$$
$$\boxed{1}$$

- $T_{\gamma_1}\circ T_{\gamma_2}\circ T_{\gamma_1}^{-1}\circ T_{\gamma_2}^{-1}$ :

$$\big[\Phi(\alpha_1),\Phi(\beta_1)\big]\big[\Phi(\alpha_2),\Phi(\beta_2)\big]\big[\Phi(\alpha_1),\Phi(\beta_1)\big]$$

Let us proceed by first computing each respective commutator in order to simplify our final commutations.

(1)

$$\text{(i)}\quad \big[\Phi(\alpha_1),\Phi(\beta_1)\big]$$
$$\text{(ii)}\quad \big[\beta_3^{-1}\beta_1^{-1}\beta_2^{-1}\beta_1\beta_3\beta_1^{-1}\beta_2\beta_1\alpha_1,\beta_1\big]$$
$$\text{(iii)}\quad \beta_3^{-1}\beta_1^{-1}\beta_2^{-1}\beta_1\beta_3\beta_1^{-1}\beta_2\beta_1\alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1}\beta_2^{-1}\beta_1\beta_3^{-1}\beta_1^{-1}\beta_2\beta_1\beta_3\beta_1^{-1}$$
$$\text{(iv)}\quad \beta_3^{-1}\beta_1^{-1}\beta_2^{-1}\beta_1\beta_3\beta_1^{-1}\beta_2\beta_1\big[\alpha_1,\beta_1\big]\beta_2^{-1}\beta_1\beta_3^{-1}\beta_1^{-1}\beta_2\beta_1\beta_3\beta_1^{-1}$$
$$\text{(v)}\quad \beta_3^{-1}\beta_1^{-1}\beta_2^{-1}\beta_1\beta_3\beta_1^{-1}\beta_2\beta_1\big[\alpha_1,\beta_1\big][\beta_2^{-1},\beta_1\beta_3^{-1}\beta_1^{-1}]$$
$$\text{(vi)}\quad \beta_3^{-1}\beta_1^{-1}\beta_2^{-1}\beta_1\beta_3\beta_1^{-1}\beta_2\beta_1\big[\alpha_1,\beta_1\big][\beta_1\beta_3^{-1}\beta_1^{-1},\beta_2^{-1}]^{-1}$$

(2)

$$\text{(i)}\quad \big[\Phi(\alpha_2),\Phi(\beta_2)\big]$$

$$\text{(ii)}\quad \big[[\beta_1\beta_3^{-1}\beta_1^{-1},\beta_2^{-1}]\alpha_2[\beta_3^{-1}\beta_1^{-1}\beta_2^{-1},\beta_1\beta_3\beta_1^{-1}]^{-1},[\beta_3^{-1}\beta_1^{-1}\beta_2^{-1},\beta_1\beta_3\beta_1^{-1}]\beta_2[\beta_3^{-1}\beta_1^{-1}\beta_2^{-1},\beta_1\beta_3\beta_1^{-1}]^{-1}\big]$$

$$\text{(iii)}\quad [\beta_1\beta_3^{-1}\beta_1^{-1},\beta_2^{-1}]\alpha_2[\beta_3^{-1}\beta_1^{-1}\beta_2^{-1},\beta_1\beta_3\beta_1^{-1}]^{-1}[\beta_3^{-1}\beta_1^{-1}\beta_2^{-1},\beta_1\beta_3\beta_1^{-1}]\beta_2$$
$$[\beta_3^{-1}\beta_1^{-1}\beta_2^{-1},\beta_1\beta_3\beta_1^{-1}]^{-1}[\beta_3^{-1}\beta_1^{-1}\beta_2^{-1},\beta_1\beta_3\beta_1^{-1}]\alpha_2^{-1}$$
$$[\beta_1\beta_3^{-1}\beta_1^{-1},\beta_2^{-1}]^{-1}[\beta_3^{-1}\beta_1^{-1}\beta_2^{-1},\beta_1\beta_3\beta_1^{-1}]\beta_2^{-1}[\beta_3^{-1}\beta_1^{-1}\beta_2^{-1},\beta_1\beta_3\beta_1^{-1}]^{-1}$$

$$\text{(iv)}\quad [\beta_1\beta_3^{-1}\beta_1^{-1},\beta_2^{-1}]\alpha_2\beta_2\alpha_2^{-1}[\beta_2^{-1},\beta_1\beta_3^{-1}\beta_1^{-1}][\beta_3^{-1}\beta_1^{-1}\beta_2^{-1},\beta_1\beta_3\beta_1^{-1}]\beta_2^{-1}[\beta_3^{-1}\beta_1^{-1}\beta_2^{-1},\beta_1\beta_3\beta_1^{-1}]^{-1}$$

$$\text{(v)}\quad [\beta_1\beta_3^{-1}\beta_1^{-1},\beta_2^{-1}]\alpha_2\beta_2\alpha_2^{-1}\beta_2^{-1}\beta_1\beta_3^{-1}\beta_1^{-1}\beta_2\beta_1\beta_3\beta_1^{-1}[\beta_3^{-1}\beta_1^{-1}\beta_2^{-1},\beta_1\beta_3\beta_1^{-1}]\beta_2^{-1}[\beta_3^{-1}\beta_1^{-1}\beta_2^{-1},\beta_1\beta_3\beta_1^{-1}]^{-1}$$

$$\text{(vi)}\quad [\beta_1\beta_3^{-1}\beta_1^{-1},\beta_2^{-1}]\big[\alpha_2,\beta_2\big]\beta_1\beta_3^{-1}\beta_1^{-1}\beta_2\beta_1\beta_3\beta_1^{-1}[\beta_3^{-1}\beta_1^{-1}\beta_2^{-1},\beta_1\beta_3\beta_1^{-1}]\beta_2^{-1}[\beta_3^{-1}\beta_1^{-1}\beta_2^{-1},\beta_1\beta_3\beta_1^{-1}]^{-1}$$

(3)

$$\text{(i)}\quad \big[\Phi(\alpha_3),\Phi(\beta_3)\big]$$

$$\text{(ii)}\quad \big[[\beta_3^{-1}\beta_1^{-1}\beta_2^{-1},\beta_1\beta_3\beta_1^{-1}]\beta_2[\beta_3^{-1}\beta_1^{-1}\beta_2^{-1},\beta_1\beta_3\beta_1^{-1}]^{-1}\beta_1\beta_3^{-1}\beta_1^{-1}\beta_2^{-1}\beta_1\beta_3\beta_1^{-1}\alpha_3[\beta_2\beta_1\beta_3,\beta_1^{-1}]^{-1}[\beta_3^{-1}\beta_1^{-1}\beta_2^{-1},\beta_1\beta_3\beta_1^{-1}]^{-1},$$
$$[\beta_3^{-1}\beta_1^{-1}\beta_2^{-1},\beta_1\beta_3\beta_1^{-1}][\beta_2\beta_1\beta_3,\beta_1^{-1}]\beta_3[\beta_2\beta_1\beta_3,\beta_1^{-1}]^{-1}[\beta_3^{-1}\beta_1^{-1}\beta_2^{-1},\beta_1\beta_3\beta_1^{-1}]^{-1}\big]$$

$$\text{(iii)}\quad [\beta_3^{-1}\beta_1^{-1}\beta_2^{-1},\beta_1\beta_3\beta_1^{-1}]\beta_2[\beta_3^{-1}\beta_1^{-1}\beta_2^{-1},\beta_1\beta_3\beta_1^{-1}]^{-1}\beta_1\beta_3^{-1}\beta_1^{-1}\beta_2^{-1}\beta_1\beta_3\beta_1^{-1}\alpha_3$$
$$[\beta_2\beta_1\beta_3,\beta_1^{-1}]^{-1}[\beta_3^{-1}\beta_1^{-1}\beta_2^{-1},\beta_1\beta_3\beta_1^{-1}]^{-1}[\beta_3^{-1}\beta_1^{-1}\beta_2^{-1},\beta_1\beta_3\beta_1^{-1}][\beta_2\beta_1\beta_3,\beta_1^{-1}]\beta_3$$
$$[\beta_2\beta_1\beta_3,\beta_1^{-1}]^{-1}[\beta_3^{-1}\beta_1^{-1}\beta_2^{-1},\beta_1\beta_3\beta_1^{-1}]^{-1}[\beta_3^{-1}\beta_1^{-1}\beta_2^{-1},\beta_1\beta_3\beta_1^{-1}][\beta_2\beta_1\beta_3,\beta_1^{-1}]$$
$$\alpha_3^{-1}\beta_1\beta_3^{-1}\beta_1^{-1}\beta_2\beta_1\beta_3\beta_1^{-1}[\beta_3^{-1}\beta_1^{-1}\beta_2^{-1},\beta_1\beta_3\beta_1^{-1}]\beta_2^{-1}$$
$$[\beta_3^{-1}\beta_1^{-1}\beta_2^{-1},\beta_1\beta_3\beta_1^{-1}]^{-1}[\beta_3^{-1}\beta_1^{-1}\beta_2^{-1},\beta_1\beta_3\beta_1^{-1}][\beta_2\beta_1\beta_3,\beta_1^{-1}]\beta_3^{-1}[\beta_2\beta_1\beta_3,\beta_1^{-1}]^{-1}[\beta_3^{-1}\beta_1^{-1}\beta_2^{-1},\beta_1\beta_3\beta_1^{-1}]^{-1}$$

$$\text{(iv)}\quad [\beta_3^{-1}\beta_1^{-1}\beta_2^{-1},\beta_1\beta_3\beta_1^{-1}]\beta_2[\beta_3^{-1}\beta_1^{-1}\beta_2^{-1},\beta_1\beta_3\beta_1^{-1}]^{-1}\beta_1\beta_3^{-1}\beta_1^{-1}\beta_2^{-1}\beta_1\beta_3\beta_1^{-1}\alpha_3\beta_3\alpha_3^{-1}\beta_3^{-1}\beta_3$$
$$\beta_1\beta_3^{-1}\beta_1^{-1}\beta_2\beta_1\beta_3\beta_1^{-1}\beta_3^{-1}\beta_1^{-1}\beta_2^{-1}\beta_1\beta_3\beta_1^{-1}\beta_2\beta_1\beta_3\beta_1\beta_3^{-1}\beta_1^{-1}\beta_2^{-1}$$
$$\beta_2\beta_1\beta_3\beta_1^{-1}\beta_3^{-1}\beta_1^{-1}\beta_2^{-1}\beta_1\beta_3^{-1}\beta_1^{-1}\beta_2\beta_1\beta_3\beta_1\beta_3^{-1}\beta_1^{-1}\beta_2^{-1}\beta_1\beta_3\beta_1^{-1}\beta_3^{-1}\beta_1^{-1}\beta_2^{-1}\beta_1\beta_3^{-1}\beta_1^{-1}\beta_2\beta_1\beta_3$$

$$\text{(v)}\quad [\beta_3^{-1}\beta_1^{-1}\beta_2^{-1},\beta_1\beta_3\beta_1^{-1}]\beta_2[\beta_3^{-1}\beta_1^{-1}\beta_2^{-1},\beta_1\beta_3\beta_1^{-1}]^{-1}\beta_1\beta_3^{-1}\beta_1^{-1}\beta_2^{-1}\beta_1\beta_3\beta_1^{-1}\big[\alpha_3,\beta_3\big]\beta_3\beta_3^{-1}\beta_1^{-1}\beta_2^{-1}\beta_1\beta_3^{-1}\beta_1^{-1}\beta_2\beta_1\beta_3$$

$$\text{(vi)}\quad [\beta_3^{-1}\beta_1^{-1}\beta_2^{-1},\beta_1\beta_3\beta_1^{-1}]\beta_2[\beta_3^{-1}\beta_1^{-1}\beta_2^{-1},\beta_1\beta_3\beta_1^{-1}]^{-1}\beta_1\beta_3^{-1}\beta_1^{-1}\beta_2^{-1}\beta_1\beta_3\beta_1^{-1}\big[\alpha_3,\beta_3\big]\beta_1^{-1}\beta_2^{-1}\beta_1\beta_3^{-1}\beta_1^{-1}\beta_2\beta_1\beta_3$$

Now that we have computed each pairs commutator, let us compute the product in order to verify that the relation holds:

– $\Phi := T_{\gamma_2}\circ T_{\gamma_2}\circ T_{\gamma_1}^{-1}\circ T_{\gamma_2}^{-1}$ :

$$\text{(i)}\quad \big[\Phi(\alpha_1),\Phi(\beta_1)\big]\big[\Phi(\alpha_2),\Phi(\beta_2)\big]\big[\Phi(\alpha_1),\Phi(\beta_1)\big]$$

$$\text{(ii)}\quad \beta_3^{-1}\beta_1^{-1}\beta_2^{-1}\beta_1\beta_3\beta_1^{-1}\beta_2\beta_1\big[\alpha_1,\beta_1\big][\beta_1\beta_3^{-1}\beta_1^{-1},\beta_2^{-1}]^{-1}[\beta_1\beta_3^{-1}\beta_1^{-1},\beta_2^{-1}]\big[\alpha_2,\beta_2\big]\beta_1\beta_3^{-1}\beta_1^{-1}\beta_2\beta_1\beta_3\beta_1^{-1}$$
$$[\beta_3^{-1}\beta_1^{-1}\beta_2^{-1},\beta_1\beta_3\beta_1^{-1}]\beta_2^{-1}[\beta_3^{-1}\beta_1^{-1}\beta_2^{-1},\beta_1\beta_3\beta_1^{-1}]^{-1}[\beta_3^{-1}\beta_1^{-1}\beta_2^{-1},\beta_1\beta_3\beta_1^{-1}]\beta_2[\beta_3^{-1}\beta_1^{-1}\beta_2^{-1},\beta_1\beta_3\beta_1^{-1}]^{-1}$$
$$\beta_1\beta_3^{-1}\beta_1^{-1}\beta_2^{-1}\beta_1\beta_3\beta_1^{-1}\big[\alpha_3,\beta_3\big]\beta_1^{-1}\beta_2^{-1}\beta_1\beta_3^{-1}\beta_1^{-1}\beta_2\beta_1\beta_3$$

$$\text{(iii)}\quad \beta_3^{-1}\beta_1^{-1}\beta_2^{-1}\beta_1\beta_3\beta_1^{-1}\beta_2\beta_1\big[\alpha_1,\beta_1\big]\big[\alpha_2,\beta_2\big]\big[\alpha_3,\beta_3\big]\beta_1^{-1}\beta_2^{-1}\beta_1\beta_3^{-1}\beta_1^{-1}\beta_2\beta_1\beta_3$$

$$\text{(iv)}\quad \beta_3^{-1}\beta_1^{-1}\beta_2^{-1}\beta_1\beta_3\beta_1^{-1}\beta_2\beta_1\beta_1^{-1}\beta_2^{-1}\beta_1\beta_3^{-1}\beta_1^{-1}\beta_2\beta_1\beta_3$$

$$\text{(v)}\quad \boxed{1}$$

6.4. **Computing** Fix($\Phi^*$).



- $A_1$ :

$$A_1 = B_3^{-1}B_1^{-1}B_2^{-1}B_1B_3B_1^{-1}B_2B_1A_1$$
$$\iff B_3^{-1}B_1^{-1}B_2^{-1}B_1B_3B_1^{-1}B_2B_1 = I$$
$$\iff B_1B_3B_1^{-1} = B_2B_1B_3B_1^{-1}B_2^{-1} \quad \text{and} \quad B_3^{-1}B_1^{-1}B_2^{-1} = B_1^{-1}B_2^{-1}B_1B_3^{-1}B_1^{-1}$$

- $A_2$ :

$$A_2 = [B_1B_3^{-1}B_1^{-1}, B_2^{-1}]A_2[B_3^{-1}B_1^{-1}B_2^{-1}, B_1B_3B_1^{-1}]^{-1}$$
$$\iff A_2^{-1}[B_1B_3^{-1}B_1^{-1}, B_2^{-1}]^{-1}A_2 = [B_3^{-1}B_1^{-1}B_2^{-1}, B_1B_3B_1^{-1}]^{-1}$$
$$\iff A_2[B_1B_3^{-1}B_1^{-1}, B_2^{-1}]A_2^{-1} = [B_3^{-1}B_1^{-1}B_2^{-1}, B_1B_3B_1^{-1}]$$

- $B_2$ :

$$B_2 = [B_3^{-1}B_1^{-1}B_2^{-1}, B_1B_3B_1^{-1}]B_2[B_3^{-1}B_1^{-1}B_2^{-1}, B_1B_3B_1^{-1}]^{-1}$$
$$\iff [[B_3^{-1}B_1^{-1}B_2^{-1}, B_1B_3B_1^{-1}], B_2] = I$$

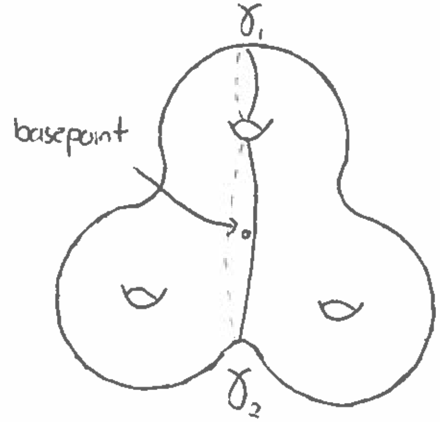
- $A_3$ :

$$A_3 = [B_3^{-1}B_1^{-1}B_2^{-1}, B_1B_3B_1^{-1}]B_2[B_3^{-1}B_1^{-1}B_2^{-1}, B_1B_3B_1^{-1}]^{-1}B_1B_3^{-1}B_1^{-1}B_2^{-1}B_1B_3B_1^{-1}$$
$$A_3[B_2B_1B_3, B_1^{-1}]^{-1}[B_3^{-1}B_1^{-1}B_2^{-1}, B_1B_3B_1^{-1}]^{-1}$$
$$A_3 = B_2B_1B_3^{-1}B_1^{-1}B_2^{-1}B_1B_3B_1^{-1}A_3[B_2B_1B_3, B_1^{-1}]^{-1}[B_3^{-1}B_1^{-1}B_2^{-1}, B_1B_3B_1^{-1}]^{-1}$$
$$A_3 = B_2B_1B_3^{-1}B_1^{-1}B_2^{-1}B_2B_1B_3B_1^{-1}B_2^{-1}A_3[B_2B_1B_3, B_1^{-1}]^{-1}[B_3^{-1}B_1^{-1}B_2^{-1}, B_1B_3B_1^{-1}]^{-1}$$
$$A_3 = B_2B_1B_3^{-1}B_1^{-1}B_2^{-1}B_2B_1B_3B_1^{-1}B_2^{-1}A_3[B_2B_1B_3, B_1^{-1}]^{-1}[B_3^{-1}B_1^{-1}B_2^{-1}, B_1B_3B_1^{-1}]^{-1}$$
$$A_3 = A_3[B_2B_1B_3, B_1^{-1}]^{-1}[B_3^{-1}B_1^{-1}B_2^{-1}, B_1B_3B_1^{-1}]^{-1}$$
$$\iff [B_2B_1B_3, B_1^{-1}]^{-1}[B_3^{-1}B_1^{-1}B_2^{-1}, B_1B_3B_1^{-1}]^{-1} = I$$
$$\iff [B_3^{-1}B_1^{-1}B_2^{-1}, B_1B_3B_1^{-1}][B_2B_1B_3, B_1^{-1}] = I$$

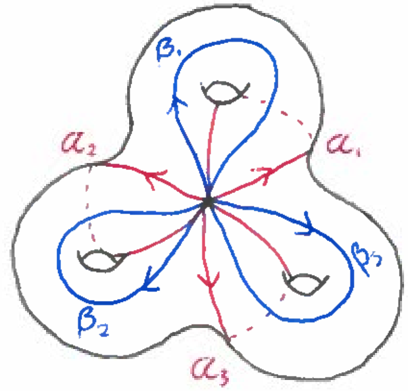
- $B_3$ :

$$B_3 = [B_3^{-1}B_1^{-1}B_2^{-1}, B_1B_3B_1^{-1}][B_2B_1B_3, B_1^{-1}]B_3[B_2B_1B_3, B_1^{-1}]^{-1}[B_3^{-1}B_1^{-1}B_2^{-1}, B_1B_3B_1^{-1}]^{-1}$$
$$= B_3$$

# Dehn twists about bounding pairs:




with basis




$T_{\gamma_1}(\alpha_1) = \alpha_1$ since $\alpha_1 \cap \gamma_1 = \emptyset$

$T_{\gamma_1}(\alpha_2) = \alpha_2$ since $\alpha_2 \cap \gamma_1 = \emptyset$

$T_{\gamma_1}(\beta_2) = \beta_2$ since $\beta_2 \cap \gamma_1 = \emptyset$

$T_{\gamma_1}(\alpha_3) = \alpha_3$ since $\alpha_3 \cap \gamma_1 = \emptyset$

$T_{\gamma_1}(\beta_3) = \beta_3$ since $\beta_3 \cap \gamma_1 = \emptyset$

~~$T_{\gamma_2}(\alpha_1) = \alpha_1$ since $\alpha_1 \cap \gamma_2 = \emptyset$~~

$T_{\gamma_2}(\alpha_2) = \alpha_2$ since $\alpha_2 \cap \gamma_2 = \emptyset$

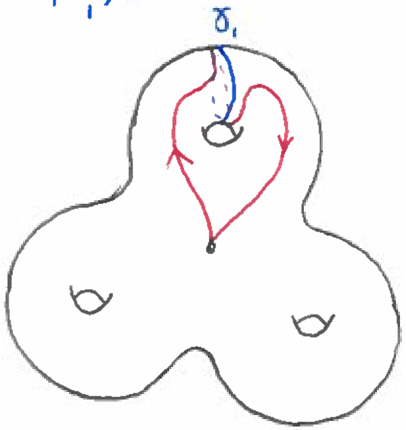
$T_{\gamma_2}(\beta_2) = \beta_2$ since $\beta_2 \cap \gamma_2 = \emptyset$

~~$T_{\gamma_2}(\alpha_3) = \alpha_3$ since $\alpha_3 \cap \gamma_2 = \emptyset$~~

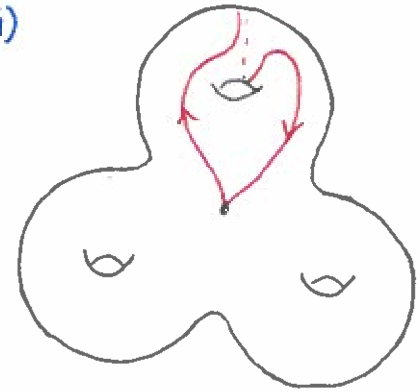
~~$T_{\gamma_2}(\beta_3) = \beta_3$ since $\beta_3 \cap \gamma_2 = \emptyset$~~

$T_{\gamma_1}(\beta_1)$:

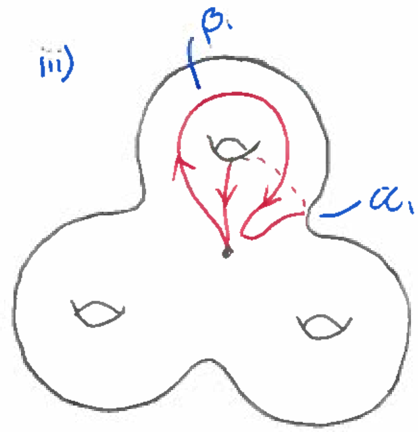
i)




ii)



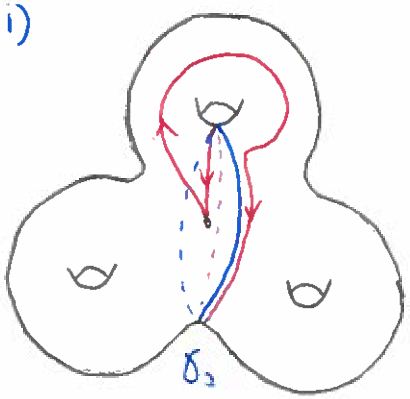
iii)



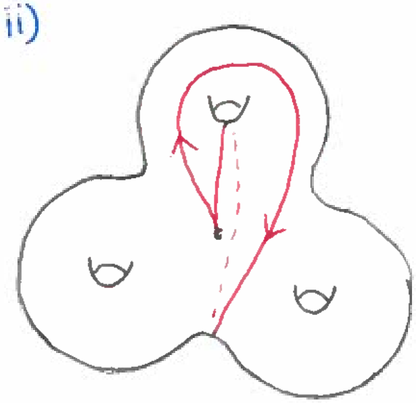

$T_{\gamma_1}(\beta_1) = \beta_1\alpha_1$

$T_{\gamma_2}(\beta_1)$:

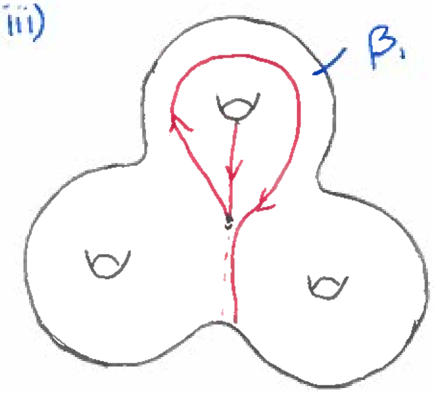
i)



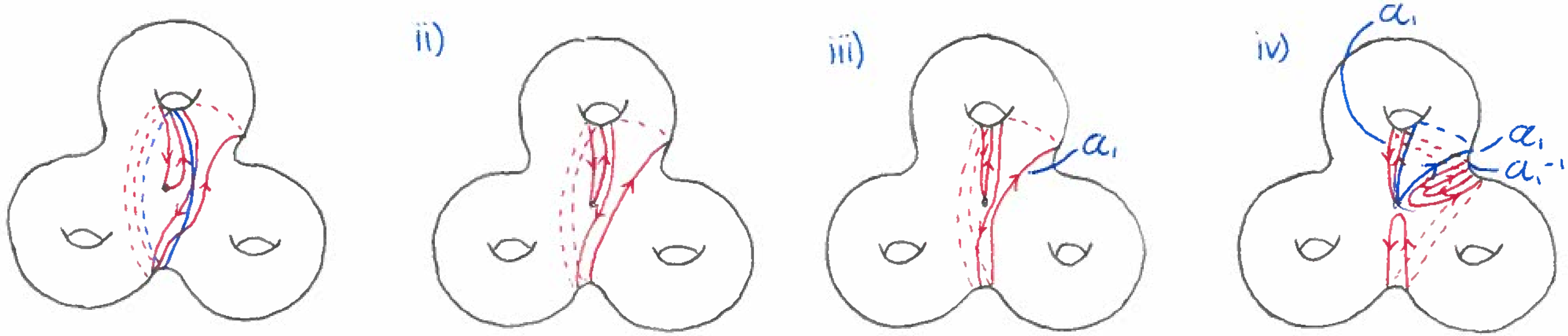

ii)

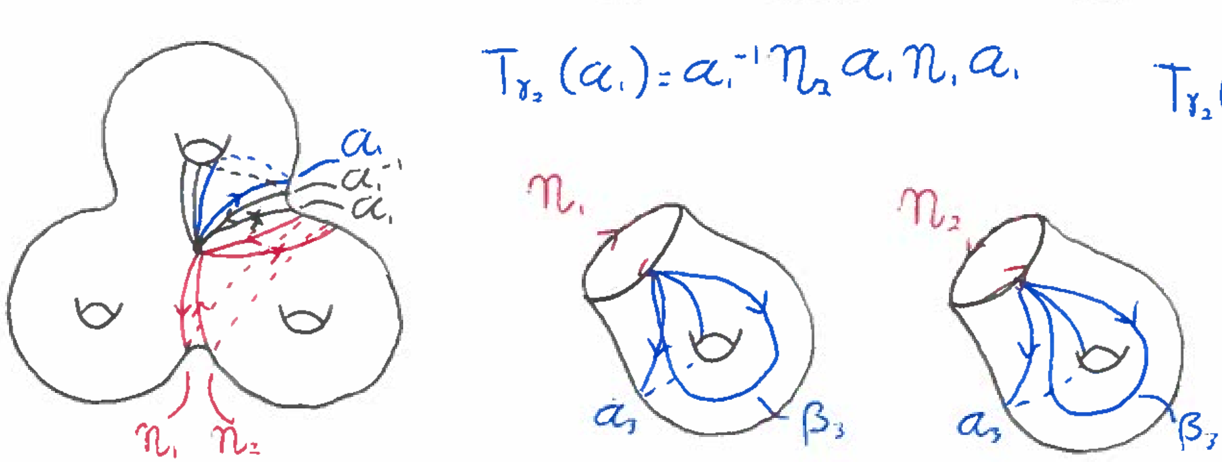


iii)



$T_{\gamma_2}(\alpha_1)$:



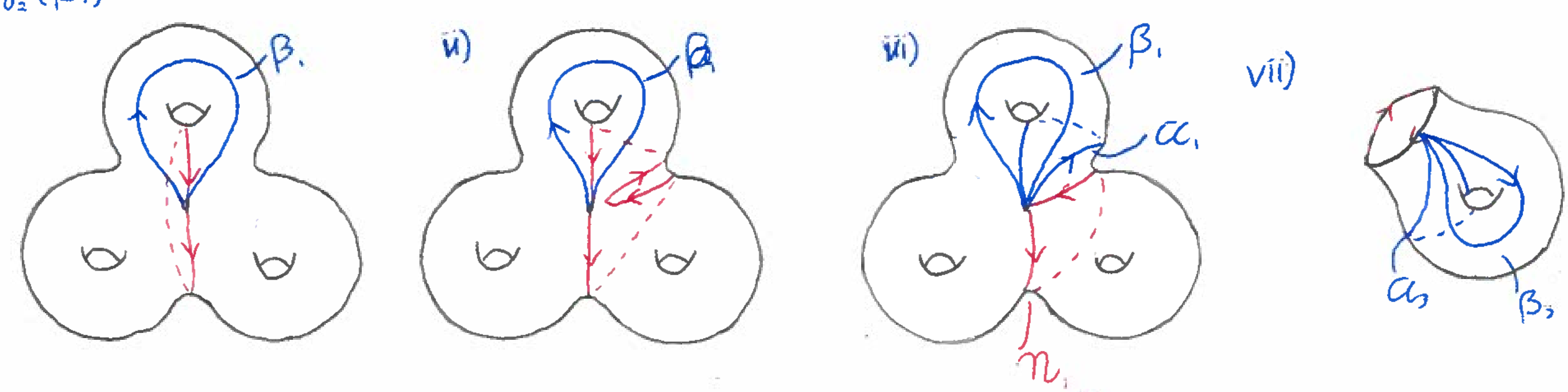




$$T_{\gamma_2}(\alpha_1) = \alpha_1^{-1}\eta_2\alpha_1\eta_1\alpha_1$$

$$T_{\gamma_2}(\alpha_1) = \alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\alpha_1[\alpha_3,\beta_3]\alpha_1$$

$$\eta_1^{-1} = \eta_2 = [\alpha_3,\beta_3]^{-1}$$

$T_{\gamma_2}(\beta_1)$:

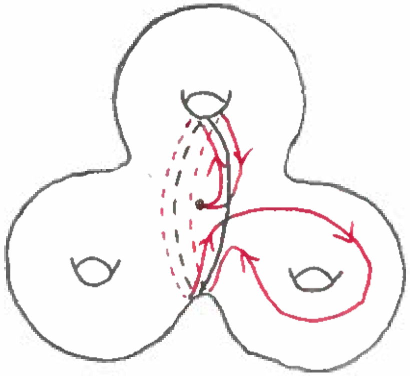



$$T_{\gamma_2}(\beta_1) = \beta_1[\alpha_3,\beta_3]\alpha_1$$

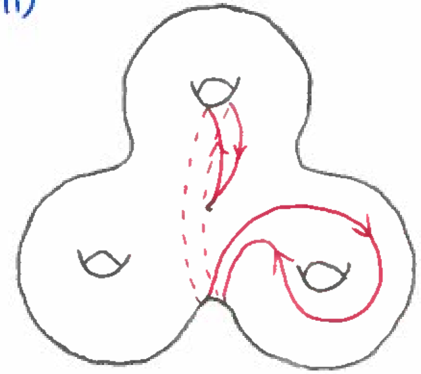
$$\eta_1 = [\alpha_3,\beta_3]$$

$T_{\gamma_2}(\beta_3)$:

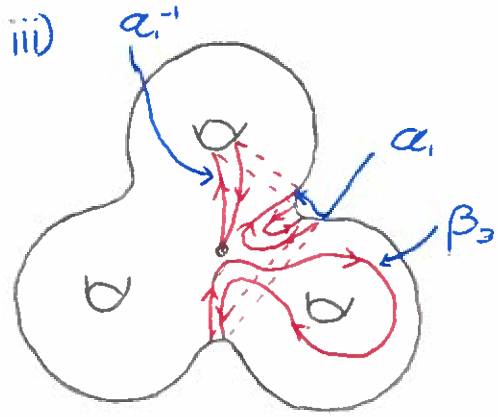
i)



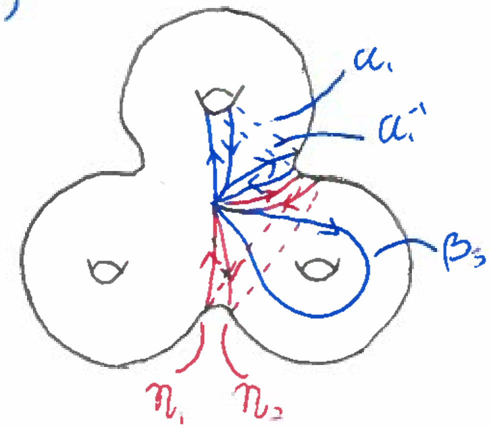
ii)



iii)

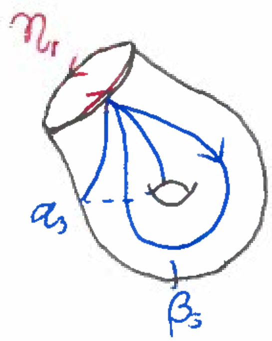


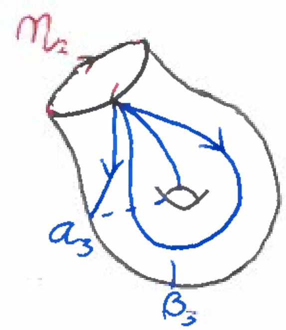

iv)




$T_{\gamma_2}(\beta_3) = \alpha_1^{-1}\eta_1\beta_3\eta_2\alpha_1$

$T_{\gamma_2}(\beta_3) = \alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\beta_3[\alpha_3,\beta_3]^{**}\alpha_1$

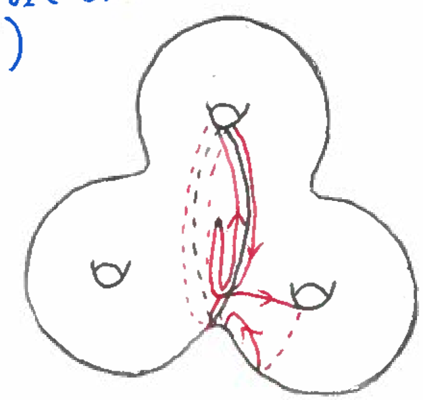


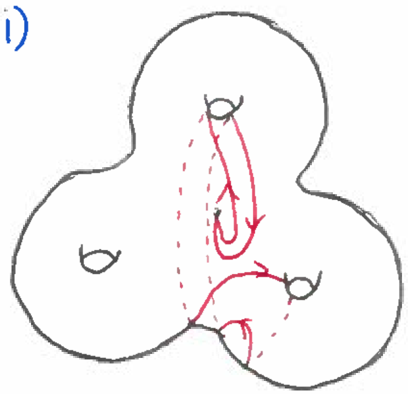

$\longrightarrow \eta_2^{-1} = \eta_1 = [\alpha_3,\beta_3]^{-1}$

$T_{\gamma_2}(\alpha_3)$:

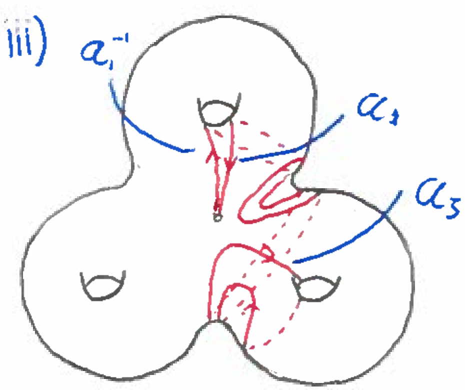
i)



ii)

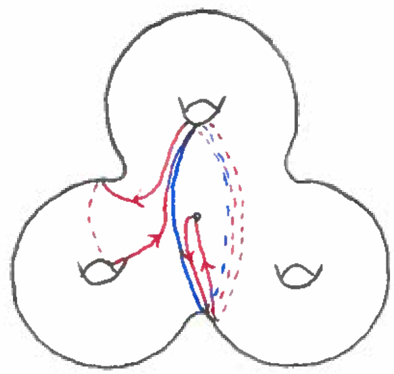


iii)

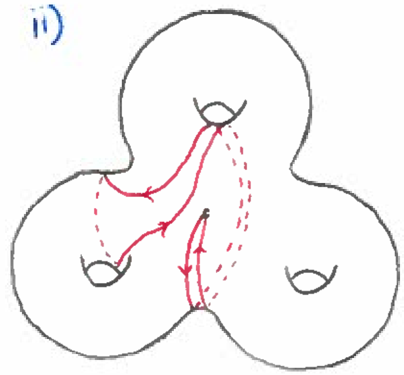



$T_{\gamma_2}(\alpha_3) = \alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\alpha_3[\alpha_3,\beta_3]^{**}\alpha_1$

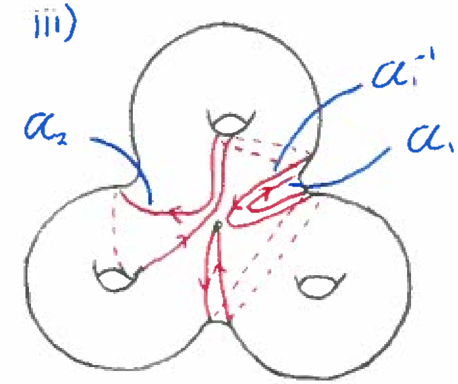
i)



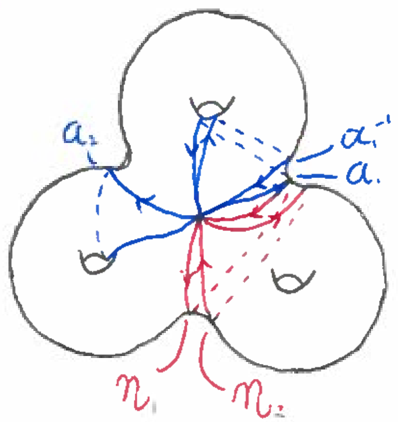
ii)



iii)

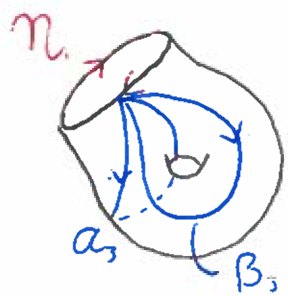


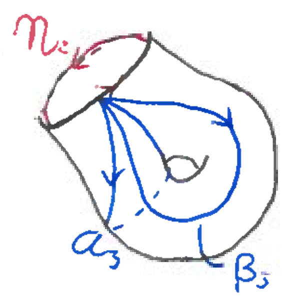

iv)




$T_{\gamma_1}(a_2) = n_1 a_1 a_2 a_1^{-1} n_2$ $\quad T_{\gamma_2}(a_2) = [a_3, \beta_3]^{-1} a_1 a_2 a_1^{-1} [a_3, \beta_3]$

$\uparrow$




$\longrightarrow n_2 = n_1^{-1} = [a_3, \beta_3]$

$T_{\gamma_1}$:

$a_1 \longmapsto a_1$ $\quad a_2 \longmapsto a_2$ $\quad a_3 \longmapsto a_3$

$\beta_1 \longmapsto \beta_1 a_1$ $\quad \beta_2 \longmapsto \beta_2$ $\quad \beta_3 \longmapsto \beta_3$

$T_{\gamma_2}$:

$a_1 \longmapsto a_1^{-1} [a_3, \beta_3]^{-1} a_1 [a_3, \beta_3] a_1$ $\quad a_2 \longmapsto a_2$ $\quad a_3 \longmapsto a_1^{-1} [a_3, \beta_3]^{-1} a_3 [a_3, \beta_3]^{-1} a_1$

$\beta_1 \longmapsto \beta_1 [a_3, \beta_3] a_1$ $\quad \beta_2 \longmapsto \beta_2$ $\quad \beta_3 \longmapsto a_1^{-1} [a_3, \beta_3]^{-1} \beta_3 [a_3, \beta_3]^{-1} a_1$

Checking Equations:

$T_{\gamma_1}: [\alpha_1, \beta_1\alpha_1][\alpha_2,\beta_2][\alpha_3,\beta_3]$

$= \alpha_1\beta_1\alpha_1\alpha_1^{-1}\alpha_1^{-1}\beta_1^{-1}[\alpha_2,\beta_2][\alpha_3,\beta_3]$

$= [\alpha_1,\beta_1][\alpha_2,\beta_2][\alpha_3,\beta_3]$

$= 1$

$T_{\gamma_2}: [\alpha_1^{-1}[\alpha_3,\beta_3]^{-1}\alpha_1[\alpha_3,\beta_3]\alpha_1, \beta_1[\alpha_3,\beta_3]\alpha_1][\alpha_2,\beta_2][\alpha_1^{-1}[\alpha_3,\beta_3]\alpha_1[\alpha_3,\beta_3]^{-1}\alpha_1, \alpha_1^{-1}[\alpha_3,\beta_3]\beta_3[\alpha_3,\beta_3]^{-1}\alpha$

$= \alpha_1^{-1}\eta^{-1}\alpha_1\eta\alpha_1\beta_1\eta\alpha_1\alpha_1^{-1}\eta^{-1}\alpha_1^{-1}\eta\alpha_1\alpha_1^{-1}\eta^{-1}\beta_1^{-1}[\alpha_2,\beta_2]\alpha_1^{-1}\eta\eta\eta^{-1}\alpha_1$

$= \alpha_1^{-1}\eta^{-1}\alpha_1\eta[\alpha_1,\beta_1][\alpha_2,\beta_2]\alpha_1^{-1}\eta\alpha_1$

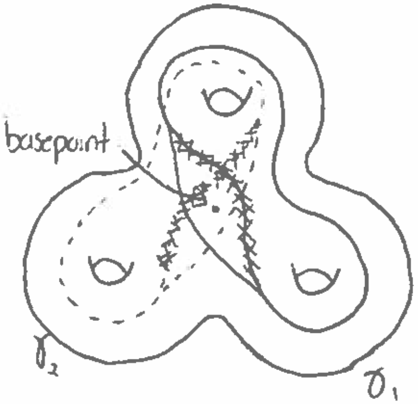
$= \alpha_1^{-1}\eta^{-1}\alpha_1\eta[\alpha_1,\beta_1][\alpha_2,\beta_2][\alpha_3,\beta_3]\eta^{-1}\alpha_1^{-1}\eta\alpha_1$

$= \alpha_1^{-1}\eta^{-1}\alpha_1\eta\eta^{-1}\alpha_1^{-1}\eta\alpha_1$

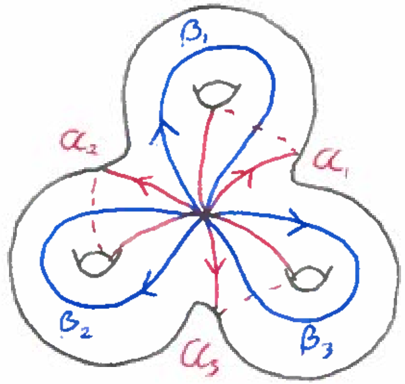
$= 1$

↑ Still works the same with corrections to $\alpha_3, \beta_3$

Dehn twists about simply intersecting curves.



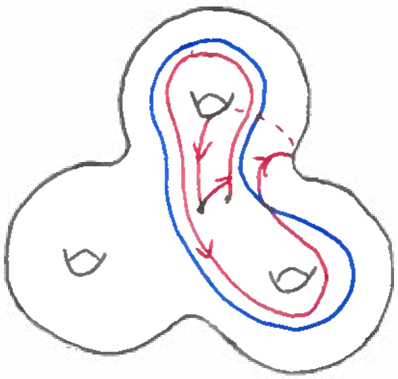

with basis



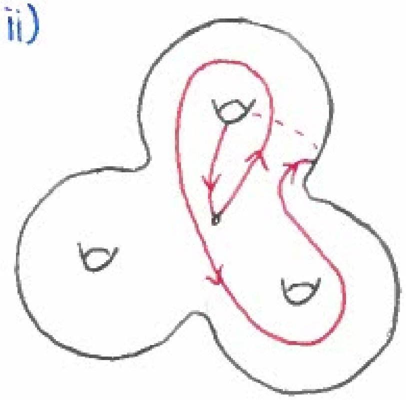

$T_{\gamma_1}(\beta_1) = \beta_1$ since $\beta_1 \cap \gamma_1 = \emptyset$
~~$T_{\gamma_1}(\alpha_2) = \alpha_2$ since $\alpha_2 \cap \gamma_1 = \emptyset$~~
~~$T_{\gamma_1}(\beta_2) = \beta_2$ since $\beta_2 \cap \gamma_1 = \emptyset$~~
$T_{\gamma_1}(\beta_3) = \beta_3$ since $\beta_3 \cap \gamma_1 = \emptyset$
$T_{\gamma_2}(\beta_1) = \beta_1$ since $\beta_1 \cap \gamma_2 = \emptyset$
$T_{\gamma_2}(\beta_2) = \beta_2$ since $\beta_2 \cap \gamma_2 = \emptyset$
~~$T_{\gamma_2}(\alpha_3) = \alpha_3$ since $\alpha_3 \cap \gamma_2 = \emptyset$~~
~~$T_{\gamma_2}(\beta_3) = \beta_3$ since $\beta_3 \cap \gamma_2 = \emptyset$~~

$T_{\gamma_1}(\alpha_1)$:

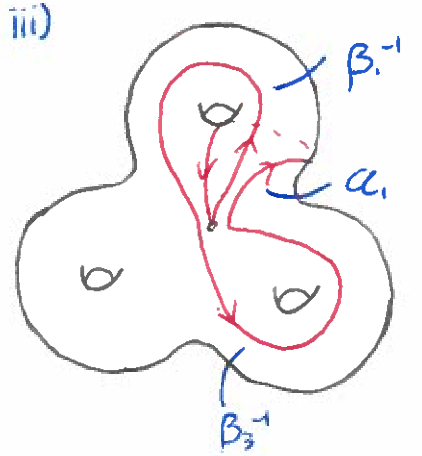
i)



ii)



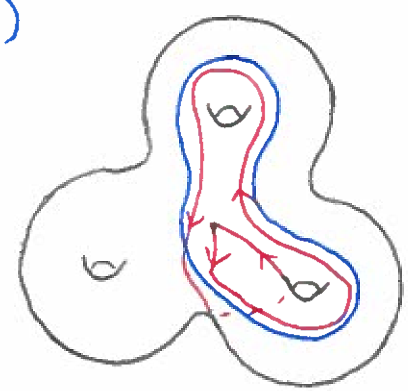
iii)



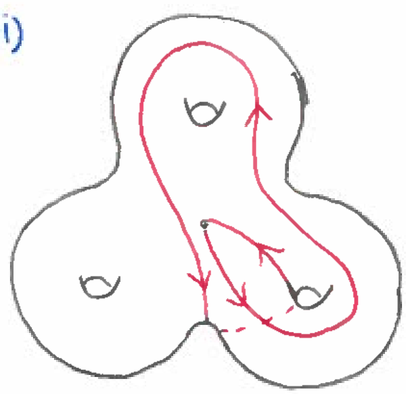

$T_{\gamma_1}(\alpha_1) = \beta_1^{-1}\beta_3^{-1}\alpha_1$

$T_{\gamma_1}(\alpha_3)$:

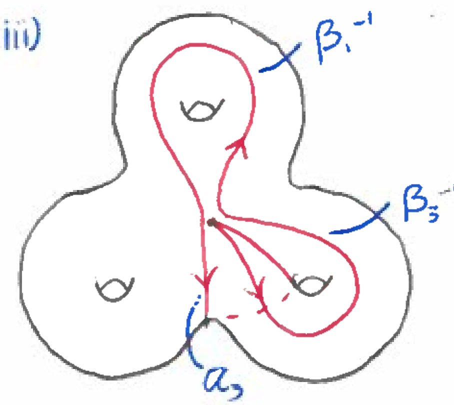
i)



ii)



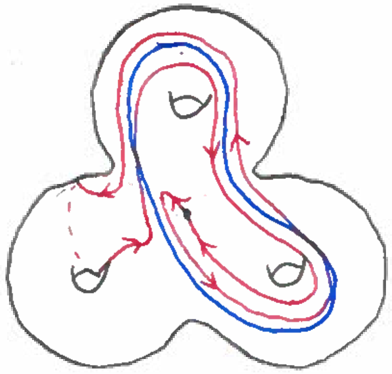
iii)



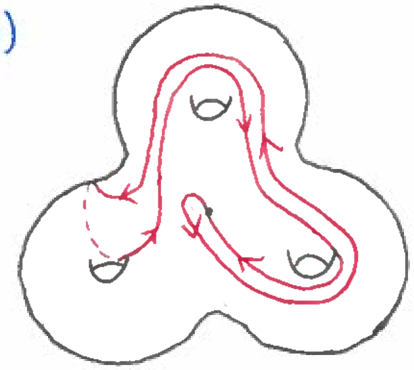

$T_{\gamma_1}(\alpha_3) = \beta_3^{-1}\beta_1^{-1}\alpha_3$

$T_{\gamma_1}(\alpha_2)$:

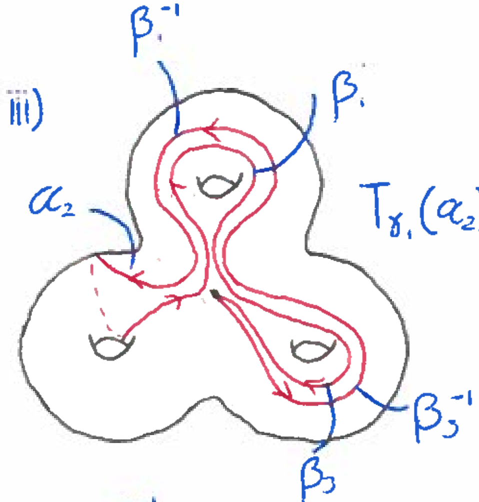
i)



ii)



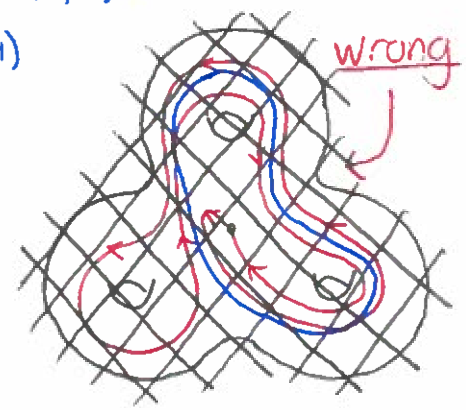
iii)



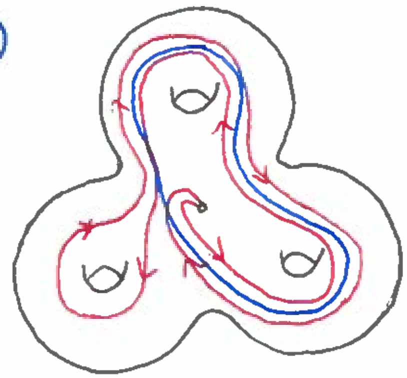

$T_{\gamma_1}(\alpha_2) = \beta_3^{-1}\beta_1^{-1}\alpha_3\beta_1\beta_5$

$T_{\gamma_1}(\beta_2)$:

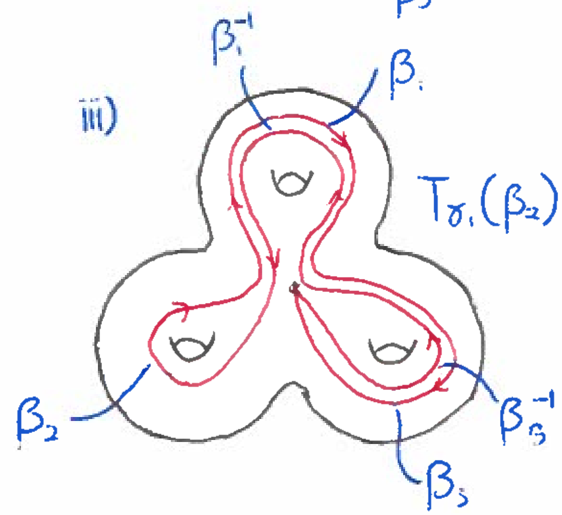
i)




ii)



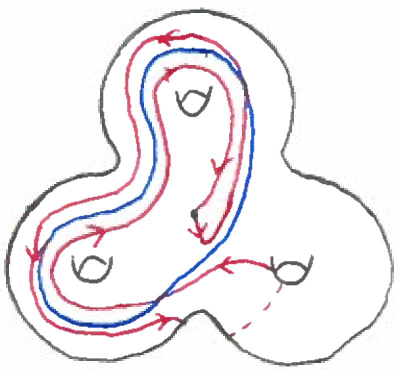
iii)



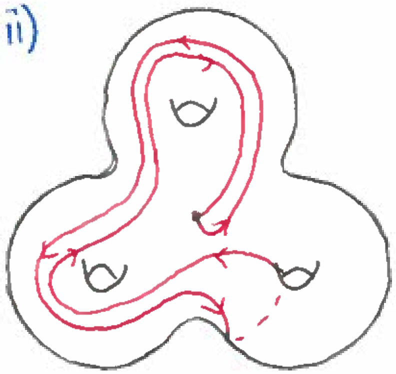

$T_{\gamma_1}(\beta_2) = \beta_3^{-1}\beta_1^{-1}\beta_2\beta_1\beta_5$

$T_{\gamma_2}(\alpha_3)$:

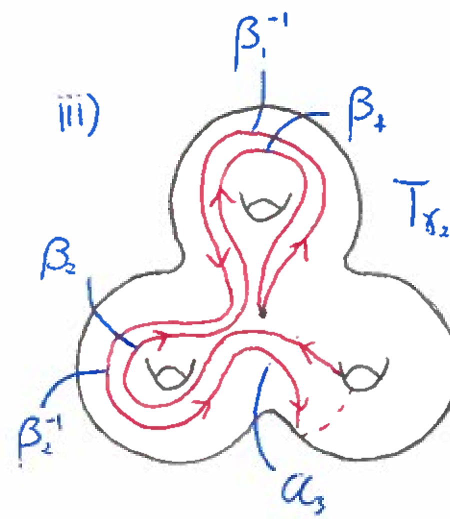
i)



ii)



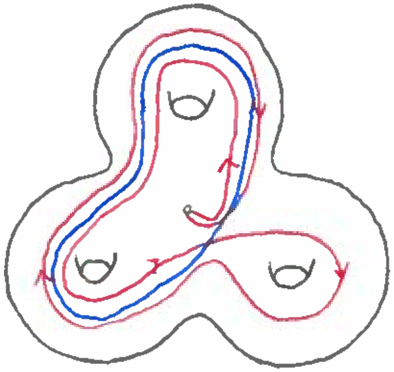
iii)



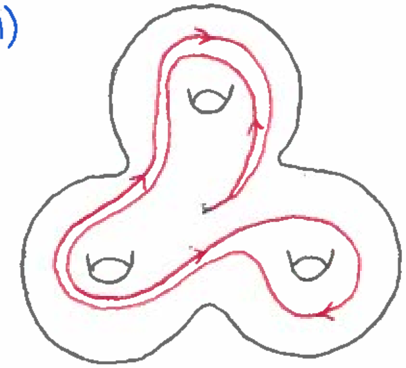

$T_{\gamma_2}(\alpha_2) = \beta_1^{-1}\beta_2^{-1}\alpha_3\beta_2\beta_1$

$T_{\gamma_2}(\beta_3)$.

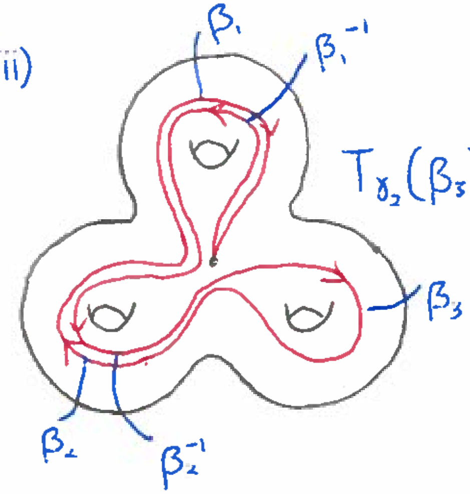
i)



ii)



iii)




$T_{\gamma_2}(\beta_3) = \beta_1^{-1}\beta_2^{-1}\beta_3\beta_2\beta_1$

Checking Equations:

$T_{\gamma_1}: [\beta_1^{-1}\beta_3^{-1}\alpha_1, \beta_1][\beta_3^{-1}\beta_1^{-1}\alpha_2\beta_1\beta_3, \beta_3^{-1}\beta_1^{-1}\beta_2\beta_1\beta_3][\beta_3^{-1}\beta_1^{-1}\alpha_3, \beta_3]$

$= \beta_1^{-1}\beta_3^{-1}\alpha_1\beta_1\alpha_1^{-1}\beta_3\beta_1\beta_1^{-1}\beta_3^{-1}\beta_1^{-1}[\alpha_2, \beta_2]\beta_1\beta_3\beta_3^{-1}\beta_1^{-1}\alpha_3\beta_3\alpha_3^{-1}\beta_1\beta_3\beta_3^{-1}$

$= \beta_1^{-1}\beta_3^{-1}\alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1}[\alpha_2, \beta_2]\alpha_3\beta_3\alpha_3^{-1}\beta_3^{-1}\beta_3\beta_1$

$= \beta_1^{-1}\beta_3^{-1}[\alpha_1, \beta_1][\alpha_2, \beta_2][\alpha_3, \beta_3]\beta_3\beta_1$

$= \beta_1^{-1}\beta_3^{-1}\beta_3\beta_1$

$= 1$

$T_{\gamma_2}: [\beta_1^{-1}\beta_2^{-1}\alpha_1, \beta_1][\beta_2^{-1}\beta_1^{-1}\alpha_2, \beta_2][\beta_1^{-1}\beta_2^{-1}\alpha_3\beta_2\beta_1, \beta_1^{-1}\beta_2^{-1}\beta_3\beta_2\beta_1]$

$= \beta_1^{-1}\beta_2^{-1}\alpha_1\beta_1\alpha_1^{-1}\beta_2\beta_1\beta_1^{-1}\beta_2^{-1}\beta_1^{-1}\alpha_2\beta_2\alpha_2^{-1}\beta_1\beta_2\beta_2^{-1}\beta_1^{-1}\beta_2^{-1}[\alpha_3, \beta_3]\beta_2\beta_1$

$= \beta_1^{-1}\beta_2^{-1}\alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1}\alpha_2\beta_2\alpha_2^{-1}\beta_2^{-1}[\alpha_3, \beta_3]\beta_2\beta_1$

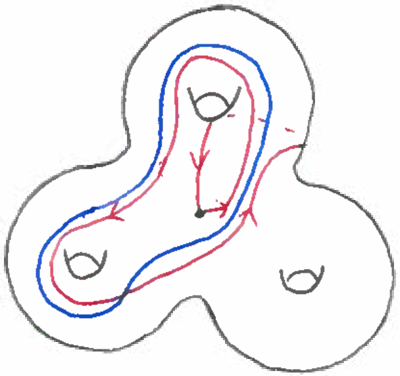
$= \beta_1^{-1}\beta_2^{-1}[\alpha_1, \beta_1][\alpha_2, \beta_2][\alpha_3, \beta_3]\beta_2\beta_1$

$= \beta_1^{-1}\beta_2^{-1}\beta_2\beta_1$

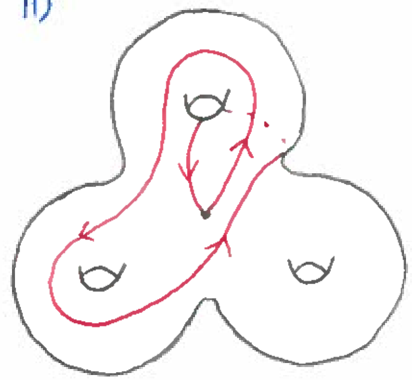
$= 1$

$T_{\gamma_2}(\alpha_1)$:

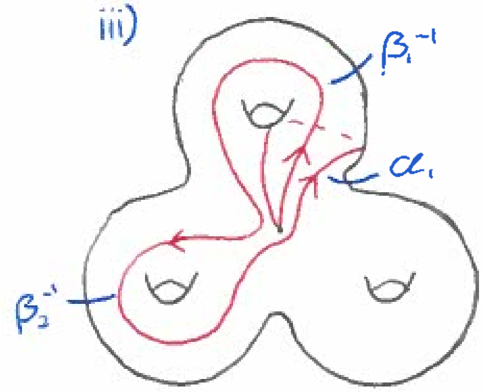
i)



ii)



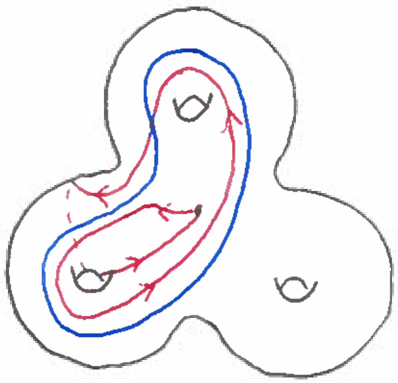
iii)

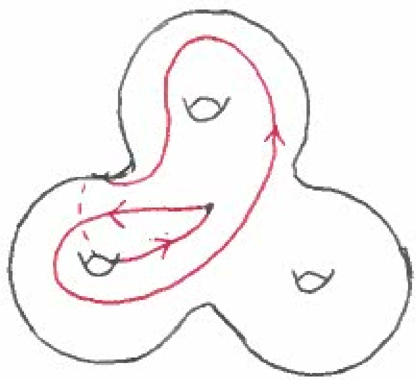


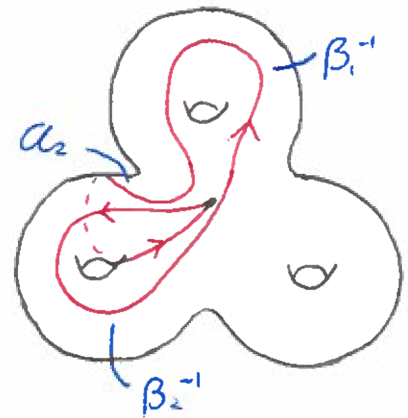

$T_{\gamma_2}(\alpha_1) = \beta_1^{-1}\beta_2^{-1}\alpha_1$

$T_{\gamma_2}(\alpha_2)$:

i)








$T_{\gamma_2}(\alpha_2) = \beta_2^{-1}\beta_1^{-1}\alpha_2$

$T_{\gamma_1}$:

$\alpha_1 \longmapsto \beta_1^{-1}\beta_3^{-1}\alpha_1$ ✓ $\quad \alpha_2 \longmapsto \beta_3^{-1}\beta_1^{-1}\alpha_2\beta_1\beta_3$ ✓ $\quad \alpha_3 \longmapsto \beta_3^{-1}\beta_1^{-1}\alpha_3$ ✓

$\beta_1 \longmapsto \beta_1$ ✓ $\quad \beta_2 \longmapsto \beta_3^{-1}\beta_1^{-1}\beta_2\beta_1\beta_3$ ✓ $\quad \beta_3 \longmapsto \beta_3$ ✓

$T_{\gamma_2}$:

$\alpha_1 \longmapsto \beta_1^{-1}\beta_2^{-1}\alpha_1$ ✓ $\quad \alpha_2 \longmapsto \beta_2^{-1}\beta_1^{-1}\alpha_2$ ✓ $\quad \alpha_3 \longmapsto \beta_1^{-1}\beta_2^{-1}\alpha_3\beta_2\beta_1$ ✓

$\beta_1 \longmapsto \beta_1$ ✓ $\quad \beta_2 \longmapsto \beta_2$ ✓ $\quad \beta_3 \longmapsto \beta_1^{-1}\beta_2^{-1}\beta_3\beta_2\beta_1$ ✓

# Technical Report

JMU Mathematics REU: Representation Varieties of 3-Manifolds

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## Contents



## 1 Character Varieties of Mapping Tori

This first section on mapping tori was written early on in the summer to help us better understand the lecture material Dr. Duncan presented at the beginning of the summer. **Some of the questions/unproven claims may be outdated with our accumulated knowledge.**

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### 1.1 Reducing $\mathcal{R}(\Sigma_\phi, G)$ to a fixed point set

Let $\Sigma$ be an orientable, connected, closed (in the manifold sense) 2-manifold. Let $\phi : \Sigma \to \Sigma$ be a homeomorphism. Define

$$\Sigma_\phi := \frac{[0,1]\times\Sigma}{\sim}, \tag{1}$$

endowed with the quotient topology of the product $[0,1]\times\Sigma$ (which has the box topology), where $\sim$ is given by

$$\begin{cases}(t,x)\sim(s,y) & \text{if } t=0, x=\phi(y), s=1,\\ (t,x)\sim(s,y) & \text{if } t=s \text{ and } x=y.\end{cases}$$

For a basepoint $x_0 \in \Sigma$, the continuous map $\phi$ induces a homomorphism (the push forward)

$$\phi_* : \pi_1(\Sigma, x_0) \to \pi_1(\Sigma, x_0). \tag{2}$$

Are we allowed to say $\Sigma$ is path connected? If so, we drop the base-point from the notation. I am aware that an open connected subset of Euclidean space is path-connected, but I'm not sure this is what we have.

**Claim: $\exists g \in \mathbb{Z}_{\geq 0}$ and $\alpha_i, \beta_i \in \pi_1(\Sigma)$ such that**

$$\pi_1(\Sigma) \cong \langle \alpha_1, \beta_1, ..., \alpha_g, \beta_g | \prod_{i=1}^{g}[\alpha_i, \beta_i]\ \rangle. \tag{3}$$

Supposedly the proof of this claim comes from the classification of surfaces and how $\pi_1$ behaves under connected sum modulo homotopy, afforded to us by Seifert van Kampen.

**Claim: The map $\iota : \Sigma \to \Sigma_\phi$ given by $x \mapsto [(0,x)]_\sim$ is a continuous injection. Therefore, $\pi_1(\Sigma)$ can be embedded into $\pi_1(\Sigma_\phi)$.** *Proof:* Let $(0,x)\sim(0,y)$. By definition of the equivalence relation, we must have $0=0$ and $x=y$ since the first entries in the order pair equal. By the above, we get an induced homomorphism $\iota_* : \pi_1(\Sigma) \to \pi_1(\Sigma_\phi)$. It is injective because Look up what Munkres says about this.

**Claim: $\exists \tau \in \pi_1(\Sigma)$ such that**

$$\begin{aligned}&\pi_1(\Sigma_\phi)\\ \cong&\langle \iota_*(\tau), \iota_*(\alpha_1), \iota_*(\beta_1), ..., \iota_*(\alpha_g), \iota_*(\beta_g) | \prod_{i=1}^{g}[\iota_*(\alpha_i), \iota_*(\beta_i)],\ \iota_*(\phi_*(\alpha_j)) = \iota_*(\tau)^{-1}\iota_*(\alpha_j)\iota_*(\tau),\ \iota_*(\phi_*(\beta_j)) = \iota_*(\tau)^{-1}\iota_*(\beta_j)\iota_*(\tau)\rangle.\end{aligned} \tag{4}$$

*Proof:* Supposedly the HNN extension.

For ease of notation, denote for all $j = 1, 2, ..., g$

$$\phi_*(\alpha_j) = Q_{j1}(\alpha_1, \alpha_2, ..., \beta_g), \tag{5}$$
$$\phi_*(\beta_j) = Q_{j2}(\alpha_1, \alpha_2, ..., \beta_g), \tag{6}$$

in which we view each $Q_{kj}$ a monomial in $2g$ indeterminates of an arbitrary group. is there a better way to specify this? We're not dealing with a ring anywhere, so...

From $\iota$ defined above, we get the map

$$\begin{aligned}\mathcal{R}(\Sigma_\phi, G) &\to \mathcal{R}(\Sigma, G)\\ \rho &\mapsto \rho\circ\iota.\end{aligned} \tag{7}$$

**Claim: This map is neither necessarily surjective nor injective.**

**Claim: If $\pi \cong \langle a_1, ..., a_n | R_1(a_1, ..., a_n), ..., R_k(a_1, ..., a_n)\rangle$ and with the discrete topology, $G$ is a topological group, and $\mathcal{R}(\pi, G)$ has the compact-open topology, then the map $F : \mathcal{R}(\pi, G) \to G^n$ given by $\rho \mapsto (\rho(a_1), ..., \rho(a_n))$, $A_i = \rho(a_i)$, is a homeomorphism onto its image of $im(F) = \{(A_1, ..., A_n) \in G^n : R_1(A_1, ..., A_n) = e, ..., R_k(A_1, ..., A_n) = e\}$. Also, the compact-open topology is the coarsest topology such that $F$ is continuous and the finest so that it is open.**

**Define *pullback* (from Duncan HW 2): Let $f : \pi \to \pi'$ be a group homomorphism of finitely presented groups. Define $f^* : \mathcal{R}(\pi', G) \to \mathcal{R}(\pi, G)$ by $\rho \mapsto \rho \circ f$.**

By using identification with $\mathcal{R}(\pi_1(\Sigma_\phi), G)$ and $\mathcal{R}(\pi_1(\Sigma), G)$ mentioned in the ereyester paragraph along with the map (7), we get a map

$$\{(T, A_1, ..., B_g) \in G^{2g+1} | \prod_{i=1}^{g}[A_i, B_i] = e, Q_{j1}(A_1, ..., B_g) = T^{-1}A_jT, Q_{j2}(A_1, ..., B_g) = T^{-1}B_jT,\ \forall j \in \{1, 2, ..., g\}\} \tag{8}$$

$$\to \{(A_1, ..., B_g) \in G^{2g} | \prod_{i=1}^{g}[A_i, B_i] = e\} \tag{9}$$

$$(T, A_1, ..., B_g) = (\rho(\iota_*(\tau)), ..., \rho(\iota_*(\beta_g))) \mapsto (\rho(\iota_*(\alpha_1)), ..., \rho(\iota_*(\beta_g))) = (A_1, ..., B_g). \tag{10}$$

It is not hard to see that the image of this map is

$$\{(A_1, ...., B_g) \in G^{2g} | \prod_{i=1}^{g}[A_i, B_i] = e, \exists T \in G \text{ s.t. } Q_{1j}(A_1, ..., B_g) = T^{-1}A_jT, Q_{2j}(A_1, ..., B_g) = T^{-1}B_jT\} \tag{11}$$

Since the set (11) is a subset of $im\ F$ where $F : \mathcal{R}(\Sigma, G) \to G^n$ is the homeomorphism mentioned above, we can to identify the set (11) with a subset of $\mathcal{R}(\Sigma, G)$. Here is the process. By surjection of $F$, $\exists \rho \in \mathcal{R}(\Sigma, G)$ such that $\rho(\alpha_i) = A_i$ and $\rho(\beta_i) = \beta_i$. Then the relation (we have $T \in G$ still) $Q_{1j}(A_1, ..., B_g) = T^{-1}A_jT$ is equivalent to $Q_{1j}(\rho(\alpha_1), ...) = T^{-1}\rho(\alpha_j)T$, which is equivalent to $\rho(Q_{1j}(\alpha_1, ..., \beta_g)) = T^{-1}\rho(\alpha_j)T$. By definition, $Q_{1j}(\alpha_1, ...) = \phi_*(\alpha_j)$. Thus, we obtain $(\rho \circ \phi_*)(\alpha_j) = T^{-1}\rho(\alpha_j)T$. Similarly, we get $(\rho \circ \phi_*)(\beta_j) = T^{-1}\rho(\beta_j)T$ for all $j = 1, ..., g$. Use the notation for pullback: $\phi^*\rho := \rho \circ \phi_*$. Note the right hand side of these equations is the evaluation of the group action defined on $\mathcal{R}(\Sigma, G)$, $g \cdot \rho \in \mathcal{R}(\pi, G)$ is defined as $x \mapsto g\rho(x)g^{-1}$. We have homomorphisms agreeing on the generators, therefore $\phi^*\rho = T^{-1} \cdot \rho$. The set (11) is identified with

$$\{\rho \in \mathcal{R}(\Sigma, G) : \exists T \in G, \phi^*\rho = T^{-1} \cdot \rho\}. \tag{12}$$

Is the following correct: If we set $T = 1$, then we obtain the fixed set of $\phi^*\rho$, which motivates step 0 in the strategy along with the reason why the fixed field on the representation variety level does not recover all the information we are interested in.

Now factoring in the action by conjugation, we get a the map (it is easy to check this is well-defined)

$$\begin{aligned} \mathcal{R}(\Sigma_\phi, G)/G &\to \mathcal{R}(\Sigma, G)/G \\ [\rho] &\mapsto [\rho \circ \iota_*], \end{aligned} \tag{13}$$

which is designed so that the following diagram commutes: $\rho \in \mathcal{R}(\Sigma_G, G) \mapsto \rho \circ \iota_* \mapsto [\rho \circ \iota_*]_\Sigma$ is the same as $\rho \mapsto [\rho]_{\Sigma_\phi} \mapsto [\rho \circ \iota_*]_{\Sigma_\phi}$. For bookkeeping (**we don't actually use this corresponding map or conjugation in this derivation**), we get the corresponding map on the images of the identification mentioned above:

$$[(T, A_1, ..., B_g)] \mapsto [(A_1, ..., B_g]$$

and that the corresponding action on the identification is $h \cdot (A_1, ..., B_g) := (hA_1h^{-1}, ..., hB_gh^{-1})$.

The point is, to find the image of the map (13), we may just find the image of the set (12) under the natural projection map onto the orbit space. In this connection, define the map $\widetilde{\phi^*} : \mathcal{R}(\Sigma, G)/G \to \mathcal{R}(\Sigma, G)/G$ by $[\rho] \mapsto [\rho \circ \phi_*] = [\phi^* \rho]$. This is well-defined by using similar logic one uses to show the map (13) is well-defined. Under the projection, the set (12) becomes $\{[\rho] : \rho \in \mathcal{R}(\Sigma, G), \exists T \in G, \phi^*\rho = T^{-1} \cdot \rho\}$. By the defining relation in the set (12), we see $\phi^*\rho \in [\rho]$. This implies that $\widetilde{\phi^*}([\rho]) = [\rho]$. Thus, the image is

$$\{[\rho] : \rho \in \mathcal{R}(\Sigma, G), \widetilde{\phi^*}([\rho]) = [\rho]\}, \tag{14}$$

which is exactly the fixed point set mentioned in part 1 of the strategy. Another note is in the special case of $T = 1$ is a subset of (12). Therefore, the image of the special case under the projection What kind of identification here? will be a subset of the image (14).

SUMMARY: we got a map $\mathcal{R}(\Sigma_\phi, G) \to \mathcal{R}(\Sigma, G)$ and want to understand what the image is. Seeing the structure directly is hard because you're dealing with abstract homomorphisms. So we transferred the map to a map of $G^{2g}$ and found the image. Then we pulled this image back to something in $\mathcal{R}(\Sigma, G)$. Using the commuting diagram mentioned above, we then found the image of this under the natural projection and saw it is a fixed point set. Now using that same commuting diagram, if we wish to find information about $\mathcal{R}(\Sigma_\phi, G)$ from this, we can directly find the fibers of $[\rho] \mapsto [\rho \circ \iota_*]$ and then find the fibers of $\rho \mapsto [\rho]$. Is this easier/harder than doing finding the fibers $\rho \circ \iota_* \mapsto [\rho]_\Sigma$ and then the fibers of $\rho \mapsto \rho \circ \iota_*$?

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### 1.2 Special case of $\Sigma = T^2, G = S^3$

Let $G = S^3$ and $\Sigma = T^2$ in the above derivations.

We may translate the condition $[\rho \circ \phi_*] =: \widetilde{\phi^*}([\rho])[\rho]$ (and thus $\exists T \in S^3, \rho \circ \phi_* = T \cdot \rho$) into an equivalent condition in $S^3 \times S^3$ using the corresponding conjugation action mentioned in the derivation. Let $A_1 = \rho(\alpha_1), B_1 = \rho(\beta_1)$ where we must have $A_1 B_1 = B_1 A_1$, since $\rho \in \mathcal{R}(T^2, S^3)$. Then we get

$$\exists T \in S^3 \text{ s.t. } (Q_{11}(A_1, B_1), Q_{22}(A_1, B_1)) = (T A_1 T^{-1}, T B_1 T^{-1}). \tag{15}$$

**Claim:** $p, q \in S^3$ **and** $pq = qp \Rightarrow p, q \in$ **span**$\langle 1, p\rangle \cap S^3$ **(an arbitrary great circle).**

**Claim:** $q \in \mathbb{H} \Rightarrow \exists p \in \mathbb{H}$ **s.t.** $pqp^{-1} \in \mathbb{C}$**. Additionally, if** $z \in$ **span**$\langle 1, p\rangle \cap S^3$**, then** $pzp^{-1} \in \mathbb{C}$**. Additionally, if** $q \in S^3$**, then** $p$ **may be taken to be in** $S^3$**.**

**Claim:** $p, q \in S^3 \cap$ **span**$\langle 1, p\rangle$ **(an arbitrary great circle) and** $\exists T \in S^3$ **s.t.** $TpT^{-1} \in S^3 \cap$ **span**$\langle 1, p\rangle$ **and** $TqT^{-1} \in S^3 \cap$ **span**$\langle 1, p\rangle$**, then** $T \in ($**span**$\langle 1, p\rangle \cap S^3) \cup ($**span**$\langle 1, p\rangle \cap S^3)j$**. Special case: if** $p, q \in S^1 \subseteq \mathbb{C}$**, then** $T \in S^1 \cap S^1 j$**.**

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## 2 Simone Project Work

We wrote a code to find a connection between Chern-Simons values with properties of the monodromy matrix of the mapping torus, specifically of the kind mentioned by Simone.

We tried to parameterize the set of monodromy matrices, but ran into technical issues with integral values.

We didn't work on the project for long as we didn't see much of a pattern. [8]

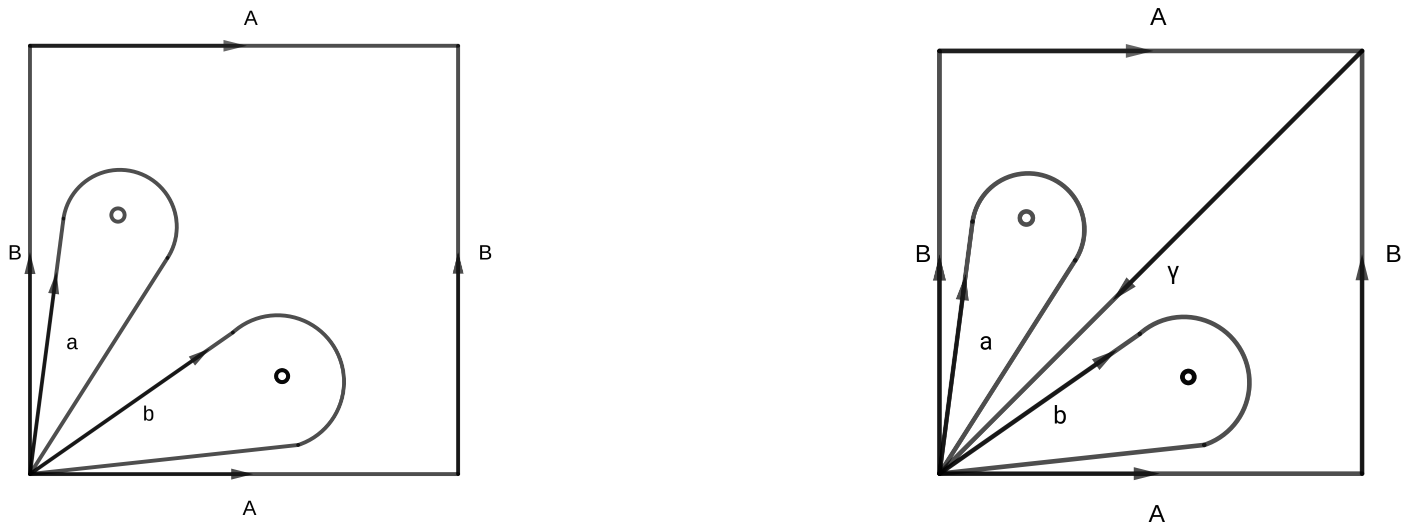


Figure 1: Two presentations of $\pi_1(T^2 \setminus \{2 \text{ pt.}\})$. Dr. Boozer used the presentation on the left, whereas we attempt to use the presentation on the right.

# 3 Boozer Project

## 3.1 Boozer Project Introduction

In [1], Dr. Boozer proved the claim that the traceless (SU(2)) character variety of the twice-punctured torus, $\chi(T^2, 2)$, is homeomorphic to $S^2 \times S^2$. He used quite advanced machinery, and he states that, "this does not seem to be easy to show from our description of this space." Such a statement gives sufficient reason to pursue an elementary proof of the homeomorphism.

Figure 1, on the left, shows the presentation of $\pi_1(T^2 \setminus \{2 \text{ pt.}\})$ of Boozer given by

$$\pi_1(T^2 \setminus \{2 \text{ pt.}\}) \cong \langle A, B, a, b \mid [A, B]ab \rangle. \tag{16}$$

We propose to use the presentation on the right of Figure 1 given by

$$\pi_1(T^2 \setminus \{2 \text{ pt.}\}) \cong \langle A, B, \gamma \rangle, \tag{17}$$

where $a = BA\gamma$ and $b = \gamma^{-1}B^{-1}A^{-1}$ (read left to right as the order in which to follow the paths). If $\rho \in \mathrm{Hom}(\pi_1(T^2 \setminus \{2 \text{ pt.}\}), \mathrm{SU}(2))$, the traceless conditions $\mathrm{tr}(\rho(a)) = \mathrm{tr}(\rho(b)) = 0$ translate to $\mathrm{tr}(\rho(\gamma)^{-1}\rho(B)^{-1}\rho(A)^{-1}) = \mathrm{tr}(\rho(B)\rho(A)\rho(\gamma)) = 0$. Using SU(2) facts, this first equation is the same as $\mathrm{tr}(\rho(A)\rho(B)\rho(\gamma)) = 0$. Therefore, using the correspondence between representation varieties and algebraic sets, we find that

$$\chi(T^2, 2) \cong \{(A, B, C) \in \mathrm{SU}(2)^3 : \mathrm{tr}(ABC) = \mathrm{tr}(BAC) = 0\}/\mathrm{Inn}(\mathrm{SU}(2)), \tag{18}$$

where the group action is diagonal conjugation by elements of SU(2).

The following subsections illustrate our attempts at realizing a homeomorphism between the space given by equation (18) and $S^2 \times S^2$. Throughout the attempts, we make extensive use of the homeomorphism $\mathrm{SU}(2) \cong S^3$.

## 3.2 Reduction to Two Variables

Put

$$E = \{(A, B) \in (S^3)^2 : \mathrm{tr}([B, A]i) = 0\}. \tag{19}$$

There is a homeomorphism between $\chi(T^2, 2)$ and $E/S^1$, the group action being conjugation by unit-length elements of the centralizer $Z(i) \cap S^3$ of $i$.

***Our Solution.*** Define the map $\phi : \chi(T^2, 2) \to E/S^1$ by $[(A, B, C)] \mapsto [(gAg^{-1}, gBg^{-1})]$, where $g \in S^3$ is any $g$ such that $gABCg^{-1} = i$.

- Perhaps one may go about this proof another way by constructing an equivariant map between $\mathcal{R}(T^2, 2)$ into $E$. However, at first glance it's not clear how to define a well-defined version in this sense. I wrote the proof below before I learned this technique, and I am not willing to spend the time to rewrite the proof.
- **Claim: $\phi$ is well-defined.** *Proof:*
  1. Such a $g$ exists since $S^3$ acts transitively on the purely imaginary quaternions.
  2. **Claim: If $g \in S^3$, $x \in \mathbb{H}^0$ (the quaternions with zero real part) with $gxg^{-1} = y$, then for $h \in S^3$, it is true that $hxh^{-1} = y \Leftrightarrow hg^{-1} \in Z(y)$, the algebra centralizer.** This claim shows that $\phi$ gives the same output for different such $g$.
  3. Let $[(A, B, C)] = [(D, E, F)]$. Then $D = hAh^{-1}, E = hBh^{-1}, F = hCh^{-1}$, and thus $DEF = hABCh^{-1}$. Let $gABCg^{-1} = i$. Then $(gh^{-1})DEF(gh^{-1})^{-1} = gh^{-1}DEFhg^{-1} = gABCg^{-1} = i$. Thus we need to check that $(gAg^{-1}, gBg^{-1}) = (gh^{-1}Dhg^{-1}, gh^{-1}Ehg^{-1})$, which is true.
  4. Let $\phi([(A, B, C)]) = (gAg^{-1}, gBg^{-1})$. By properties of trace, $0 = \mathrm{tr}(BAC) = \mathrm{tr}(gBACg^{-1}) = \mathrm{tr}(gBg^{-1}gAg^{-1}gCg^{-1})$. Using $gABCg^{-1} = i$, we receive that $gCg^{-1} = gB^{-1}A^{-1}g^{-1}i$. Combining these two, we see that $0 = \mathrm{tr}(gBg^{-1}gAg^{-1}gB^{-1}A^{-1}g^{-1}i) = \mathrm{tr}(gBAB^{-1}A^{-1}g^{-1}i) = \mathrm{tr}([B, A]i)$. **Thus,** $\mathrm{range}(\phi) \subseteq E$.
- **Claim: $\phi$ is surjective.** *Proof:* Put $A, B \in S^3$ such that $\mathrm{tr}([B, A]i) = 0$ and let $C = B^{-1}A^{-1}i$. Consider $\mathrm{tr}(ABC) = \mathrm{tr}(ABB^{-1}A^{-1}i) = \mathrm{tr}(i) = 0$. Also note that $\mathrm{tr}(BAC) = \mathrm{tr}(BAB^{-1}A^{-1}i) = \mathrm{tr}([B, A]i) = 0$ by hypothesis. Then $\phi([(A, B, C)]) = (A, B)$, and since $A, B$ were arbitrary elements in $E$, the map is surjective.
- **Claim: $\phi$ is injective.** *Proof:* Let $\phi([(A, B, C)]) = \phi([(D, E, F)])$. Then $(gAg^{-1}, gBg^{-1}, gCg^{-1}) = (hDh^{-1}, hEh^{-1}, hFh^{-1})$ for some $g, h \in S^3$. Hence, $(D, E, F) = (h^{-1}gAg^{-1}h, h^{-1}gBg^{-1}h, h^{-1}gCg^{-1}h)$. Thus, $(D, E, F) \sim (A, B, C)$, which implies $[(A, B, C)] = [(D, E, F)]$.
- **Claim $\phi$ is continuous.** *Proof:* The map $F \to E$ given by $(A, B, C) \mapsto (gAg^{-1}, gBg^{-1})$ consists of multiplications, inverse, (cartesian) product map, which are all continuous. This map is continuous. **Claim: the projection map onto the orbit space is a quotient map.** Corollary: $\phi$ is continuous by Theorem 22.2 from Munkres. The pre-image given by the third bullet point above along with composition with the projection mapping is continuous, so the inverse is continuous. **Q: orbit space compact by a theorem?**

■

The natural next step is to write $E$ as a union of fibers of the trace of the commutator over $\mathrm{span}\langle 1, j, k\rangle \cap S^3$. However, the fibers are not all homeomorphic, which makes it hard to see the global topology. Moreover, $\mathrm{tr}([A, B]i)$ cannot be decomposed into $\mathrm{tr}(A), \mathrm{tr}(B), \mathrm{tr}(C), \mathrm{tr}(AB), \mathrm{tr}(AC), \mathrm{tr}(BC)$ only. We abandoned this approach for these reasons. **Although, if we are not mistaken all the fibers, except of 1, are homeomorphic. Thus if one could analyze how the fibers hook up at 1, one may be able to revive this approach. We did not do this, but it seems possible.**

We also tried looking at map from $E$ to $S^3$ given by $(A, B) \mapsto AB$. We attempted to compute the fibers of this map by working out cases on domain, which was a bad technique. **We never re-attempted this method with this reflection, but someone should.**

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### 3.3 Failed Attempt

NOTE: This Section 3.3 contains a bunch of nonsense in it (e.g. mistaking a fiber bundle for a global product structure. I considered deleting it, but I wanted to comb through it for any seeds. Claim: $E/S^1 \cong S^2 \times S^2$.

***Our Solution.*** Define $f : S^3 \times S^3 \to S^3$ as $(A, B) \mapsto [B, A]$. It is straightforward to see that

$$E = \bigcup_{X \in \text{span}\langle 1,j,k\rangle \cap S^3} f^{-1}(X).$$

Let $(A, B) \in E$. Then $\exists! X = \cos(\alpha) + I_\alpha(X)\sin(\alpha) = \cos(\alpha) + I_{-\alpha}(X)\sin(-\alpha) \in S^3$ such that $f(A, B) = X$. Define the equivalence relation $e^{i\alpha} \sim_1 e^{-i\alpha}$. We get a well-defined surjection from $\text{span}\langle 1, j, k\rangle \cap S^3$ to $S^1/\sim_1$ by $X \mapsto [e^{i\alpha}]_{\sim_1}$ (extending to the case of $X = \pm 1$ by having it map to itself).

Map $f^{-1}(X) \to f^{-1}(-j) = \{(C, D) \in (S^3)^2 : [C, D] = -j\}$ via

$$(A, B) \mapsto (gBg^{-1}, gAg^{-1}) = (-gB(-g)^{-1}, -gA(-g)^{-1}),$$

where $g \in Z(i)$ such that $g[B, A]g^{-1} = j$. This is a homeomorphism.

Analyze $f^{-1}(-j) = \{(a, b) \in (S^3)^2 : a, \bar{b}, a\bar{b} \perp 1 + j\}$. Write $a = a_1 + a_2 i + a_3 j + a_4 k, b = b_1 + b_2 i + b_3 j + b_4 k$. Then we see that $(a, b) \in f^{-1}(-j) \Leftrightarrow a_1 = -a_3, b_1 = b_3$ (i.e. $a, b$ are practically in $\mathbb{R}^3$) and

$$(b_2 + b_4)a_2 + 2b_3 a_3 + (b_4 - b_2)a_4 = 0. \tag{20}$$

Since $b \in S^3$, we see that $b_2^2 + (\sqrt{2}b_3)^2 + b_4^2 = 1$, and we may parameterize ($1-1$ homeomorphism) this ellipsoid as $b_2 = \sin(\theta)\sin(\varphi), b_3 = \frac{1}{\sqrt{2}}\sin(\theta)\sin(\varphi), b_4 = \cos(\theta)$. This will make equation (20) describe a plane passing through the origin in the variables $a_2, a_3, a_4$. Enforcing that $a \in S^3$ (and thus $(a_2, a_3, a_4)$ lies on the same ellipsoid as $(b_2, b_3, b_4)$), we see that $(a_2, a_3, a_4)$ must consequently lie on a great ellipse determined by $b$. Do a coordinate transformation to turn this into a great circle. Every quaternion $y$ on this great circle is given by $y = \cos(\omega) + I_\omega(y)\sin(\omega) = \cos(\omega) + I_{-\omega}(y)\sin(-\omega)$ (if it turns out the great circle is perpendicular to 1, then ). Define the equivalence relation $\sim_2$ as $e^{i\omega} \sim_2 e^{-i\omega}$. We get a $1-1$ map from that great circle to $S^1/\sim_2$ by $y \mapsto [e^{i\omega}]$.

Map the original $(A, B)$ to the tuple

$$([e^{i\alpha}]_{\sim_1}, [e^{i\omega}]_{\sim_2}, (\sin(\theta)\cos(\varphi), \frac{1}{\sqrt{2}}\sin(\theta)\sin(\varphi), \cos(\theta))) \in S^1/\sim_1 \times S^1/\sim_2 \times S^2 \cong (S^1 \times S^1)/(\sim_1 \times \sim_2) \times S^2 \cong S^2 \times S^2.$$

**Claim: This map is conjugation invariant under the $Z(i)$ action.** *Proof:* Let $g \in Z(i)$. First, we see that $f(gAg^{-1}, gBg^{-1}) = gf(A, B)g^{-1}$. The corresponding $X$'s are $X$ vs. $gXg^{-1}$. We see that $\text{Re}(X) = \text{Re}(gXg^{-1})$, so the first coordinates of the map are the same. Further, if $h \in Z(i)$ such that $hXh^{-1} = j$, then $(hg^{-1})(gXg^{-1})(hg^{-1})^{-1} = j$. Thus the corresponding maps $f^{-1}(X) \to f^{-1}(j)$ vs. $f^{-1}(gXg^{-1}) \to f^{-1}(-j)$ are defined by the rule $(A, B) \mapsto (hBh^{-1}, hAh^{-1})$ vs. the rule $(A, B) \mapsto (hg^{-1}A(hg^{-1})^{-1}, hg^{-1}B(hg^{-1})^{-1})$. It is seen that the in the first map (looking at the images of the specific points now), $(A, B) \mapsto (hBh^{-1}, hAh^{-1})$ and in the second map $(gAg^{-1}, gBg^{-1}) \mapsto (hBh^{-1}, hAh^{-1})$, the same thing. Hence, the second and third entries in the tuple of the map will be the same. ■

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### 3.4 Trace Map and Identities

After putting the reduction to two variables approach aside, we realized the importance of looking at conjugation-invariant maps out of $\mathcal{R}(T^2, 2)$. For example, a trace invariant map from $\mathcal{R}(T^2, 2)$ induces a well-defined map from $\chi(T^2, 2)$. Another reason, which we did not really apply, is that of the niceness of principal bundles, for which one must have a projection map whose fibers are preserved by the diagonal conjugation action.

For three variables in $\mathrm{SL}(2,\mathbb{C})$, the most general and natural such map is $\Phi : \mathrm{SL}(2,\mathbb{C})^3 \to \mathbb{C}^6$ given by

$$(A,B,C) \mapsto (\mathrm{tr}(A), \mathrm{tr}(B), \mathrm{tr}(C), \mathrm{tr}(AB), \mathrm{tr}(AC), \mathrm{tr}(BC)). \tag{21}$$

The reason this is the most general such map is that an extension of the classical result of Fricke and Vogt states that $\Phi$ is surjective [5]. Moreover, trace of any monomial of $\mathrm{SL}(2,\mathbb{C}$ matrices can be expressed as a polynomial with integer coefficients in these 6 traces. See [10], [9], [2], [7] for related information and the two variables case. See [12] and [11] for information on generators of ring of invariants. The source [12] is particularly general.

The particular relations for $\mathrm{tr}(ABC)$ and $\mathrm{tr}(BAC)$ are

$$\mathrm{tr}(ABC) = P(T) := t_1t_{23} + t_2t_{13} + t_3t_{12} - t_1t_2t_3, \tag{22}$$

$$\mathrm{tr}(BAC) = Q(T) := t_1^2 + t_2^2 + t_3^2 + t_{12}^2 + t_{13}^2 + t_{23}^2 + t_{12}t_{13}t_{23} - t_1t_2t_{12} - t_1t_3t_{13} - t_2t_3t_{23} - 4, \tag{23}$$

where $t_1 = \mathrm{tr}(A), t_2 = \mathrm{tr}(B), t_3 = \mathrm{tr}(C), t_{12} = \mathrm{tr}(AB), t_{13} = \mathrm{tr}(AC), t_{23} = \mathrm{tr}(BC)$, and $T = (t_1, t_2, t_3, t_{12}, t_{13}, t_{23})$. By slight abuse of notation, define

$$\mathcal{A} = \{T \in [-2,2]^6 : P(T) = Q(T) = 0\}. \tag{24}$$

When restricted, $\mathrm{SU}(2)^3$, $\Phi|_{\mathrm{SU}(2)^3}$, maps into $[-2,2]^6 \subseteq \mathbb{R}^6$. By slight abuse of notation, henceforth denote the further restriction $\Phi|_{\mathcal{R}(T^2,2)} : \mathcal{R}(T^2,2) \to \mathcal{A}$ as $\Phi$. As noted above, $\Phi$ induces a map from $\chi(T^2,2)$, henceforth denoted as $\tilde{\Phi}$.

**Lemma (Moment Problem): Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ and let $c_1, c_2 \in \mathbb{R}$. Then there exists a unique vector** $\mathbf{c} \in \mathbf{span}\langle \mathbf{a}, \mathbf{c} \rangle$ such that $\mathbf{a} \cdot \mathbf{c} = c_1$ and $\mathbf{b} \cdot \mathbf{c} = c_2$.

**Claim: $\tilde{\Phi} : \chi(T^2,2) \to \mathcal{A}$ is injective.**

***Our Solution.*** Let $\Phi(A,B,C) = \Phi(D,E,F)$. Then $\mathrm{tr}(A) = \mathrm{tr}(D), \mathrm{tr}(B) = \mathrm{tr}(E), \mathrm{tr}(C) = \mathrm{tr}(F), \mathrm{tr}(AB) = \mathrm{tr}(DE), \mathrm{tr}(AC) = \mathrm{tr}(DF), \mathrm{tr}(BC) = \mathrm{tr}(EF)$. Write $A = r + \mathbf{u}, B = s + \mathbf{v}, C = t + \mathbf{w}, D = r' + \mathbf{u}', E = s' + \mathbf{v}', F = t' + \mathbf{w}'$. From the first three equalities we immediately receive $r = r', s = s', t = t'$ and $||\mathbf{u}|| = ||\mathbf{u}'||$, $||\mathbf{v}|| = ||\mathbf{v}'||$, $||\mathbf{w}|| = ||\mathbf{w}'||$. The last three equalities from the injectivity hypothesis state $rs - \mathbf{u} \cdot \mathbf{v} = r's' - \mathbf{u}' \cdot \mathbf{v}', rt - \mathbf{u} \cdot \mathbf{w} = r't' - \mathbf{u}' \cdot \mathbf{w}', st - \mathbf{v} \cdot \mathbf{w} = s't' - \mathbf{v}' \cdot \mathbf{w}'$. These imply that $\mathbf{u} \cdot \mathbf{v} = \mathbf{u}' \cdot \mathbf{v}', \mathbf{u} \cdot \mathbf{w} = \mathbf{u}' \cdot \mathbf{w}', \mathbf{v} \cdot \mathbf{w} = \mathbf{v}' \cdot \mathbf{w}'$. Further note from equation (31) below that $\mathbf{u}, \mathbf{v}, \mathbf{w}$ span a two-dimensional vector space. By transitivity of conjugation, $\exists g \in S^3$ such that $g(r + \mathbf{u})g^{-1} = r + \mathbf{u}'$. **Claim (*): $\exists h \in Z(\mathbf{u}')$ such that $hg(s + \mathbf{v})(hg)^{-1} = s + \mathbf{v}'$.** One way to see this is once $\mathbf{u}$, $\mathbf{u} \cdot \mathbf{v}$, and a plane containing $\mathbf{0}, \mathbf{u}$ are all fixed, then there are only two choices in that plane for a possible $\mathbf{v}$. These choices are conjugate to each other via an element of $Z(\mathbf{u})$ (specifically a reflection of the plane). Recall that $\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{0}$ lie in a plane, and $\mathbf{u}', \mathbf{v}', \mathbf{w}', \mathbf{0}$ lie in a plane. When $\mathbf{u}$ is conjugated to equal $\mathbf{u}'$, we further conjugate via $\ell \in Z(\mathbf{u}')$ so that the plane spanned by $\ell g\mathbf{u}(\ell g)^{-1}, \ell g\mathbf{v}(\ell g)^{-1}, \ell g\mathbf{w}(\ell g)^{-1}$ coincides with the plane spanned by $\mathbf{u}', \mathbf{v}', \mathbf{w}'$. Since dot products are invariant under this conjugation (rotation), we find that $g\mathbf{v}g^{-1}$ and $\mathbf{v}'$ must be one of two possible solutions, which are conjugate to each other via an element $h' \in Z(g\mathbf{u}g^{-1}) = Z(\mathbf{u}')$. Therefore, put $h = h'\ell g$ so that $\mathbf{u}' = h\mathbf{u}h^{-1}, \mathbf{v}' = h\mathbf{v}h^{-1}$. Claim (*) is shown. To finish the proof, by uniqueness of the Moment Problem (see above lemma), we get that $\mathbf{w}' = h\mathbf{w}h^{-1}$. ■

What is the image $\tilde{\Phi}$? In section 3.6, we show $\tilde{\Phi}$ is not surjective. In some ways this is the main difficulty of the problem. If one identifies the image, then we believe the complete solution to the project would follow. The difficult part is trying to express $\mathrm{SU}(2)$ conditions (e.g. orthonormality, determinant 1) via the trace map.

**Something we're not sure we spent enough time thinking about is whether the result of Procesi [12], specifically that of the specification of the generators for the ring of invariants of $\mathrm{U}(2)$ invariants, would be useful in determining the image. One would have find an analogous result for $\mathrm{SU}(2)$ invariants.**

### 3.5 Goldman Papers

Theorem 4.3 of Goldman [4] states that $\{(x, y, z) \in (-2, 2)^3 : x^2 + y^2 + z^2 - xyz - 2 < 2\} \subseteq$ *image* $f$, where $f : \mathrm{SU}(2)^2 \to \mathbb{R}^3$ is given by $f(A, B) = (\mathrm{tr}(A), \mathrm{tr}(B), \mathrm{tr}(AB))$. This result is useful in determining the image of $f$ (one need only further work out the edge cases). We tried to extend this result and combine it with Goldman's 3-variable $\mathrm{SL}(2, \mathbb{C})$ case result (see section 5 in [5]) to get the 3-variable SU(2) analog. We could not extend the proof of his Theorem 4.3 because we attempted to find a particular solution in SU(2) to the map (21) using the particular $\mathrm{SL}(2, \mathbb{C})$ solutions Goldman provided on page 574 of [4]. We obtained a solution but needed to satisfy a large inequality. **However, we prematurely stopped pursuing this method for no apparent reason. Moreover, as we reflect on this approach with the added knowledge gained from our other attempts, we see potential in reviving this approach.**

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### 3.6 Non-surjectivity Result

$\Phi : \mathcal{R}(T^2, 2) \to \mathcal{A}$, as defined in Section 3.4, is not surjective.

***Our Solution.*** Let $T = (t_1, t_2, t_3, t_{12}, t_{13}, t_{23}) \in \mathcal{A}$. Assume that $\exists (A, B, C) \in \mathcal{R}(T^2, 2)$ such that $\Phi(A, B, C) = T$. Write $A = r + \mathbf{u}, B = s + \mathbf{v}, C = t + \mathbf{w}$. We require $s = \frac{t_2}{2}$, $t = \frac{t_3}{2}$, and $\frac{t_{23}}{2} = st - \mathbf{v} \cdot \mathbf{w}$. Therefore, we require

$$\cos(\theta) = \frac{\frac{t_2 t_3}{4} - \frac{t_{23}}{2}}{\sqrt{\left(1 - \frac{t_2^2}{4}\right)\left(1 - \frac{t_3^2}{4}\right)}}, \tag{25}$$

where $\theta$ is the relative angle between $\mathbf{v}$ and $\mathbf{w}$, which implies

$$-1 \leq \frac{\frac{t_2 t_3}{4} - \frac{t_{23}}{2}}{\sqrt{\left(1 - \frac{t_2^2}{4}\right)\left(1 - \frac{t_3^2}{4}\right)}} \leq 1. \tag{26}$$

However, notice that

$$(t_1, t_2, t_3, t_{12}, t_{13}, t_{23}) = (1.95778975, 1.52734115, -1.5, 1.81, -1.8, 0.5) \in \mathcal{A}$$

(within a $10^{-8}$ error) but that

$$\frac{\frac{t_2 t_3}{4} - \frac{t_{23}}{2}}{\sqrt{\left(1 - \frac{t_2^2}{4}\right)\left(1 - \frac{t_3^2}{4}\right)}} = -1.92669413696. \tag{27}$$

■

**We wonder whether the inequalities given by the cosines of relative angles are sufficient to determine the image. One should relate these inequalities to the inequalities of Sylvester's Theorem below and perhaps to the inequality mentioned in Section 3.5.**

---

### 3.7 Find the Image

The main crux of the problem is to find the image of this trace map. We expect that it is the above variety but with closed inequality constraints enforced. The issue with just using the Fricke-Vogt equations is that only encodes $\mathrm{SL}(2, \mathbb{C})$ information and says nothing about SU(2). The additional information we must incldue is that determinant $= 1$ and that the columns form an orthonormal basis.

There are a few equivalent perspectives one may take in attempting to encode this information

- $S^3$ Scalar + Vector Approach
  - Viewing the scalar and 3D vector as separate objects but related to one another via the unit modulus condition.
  - View as an indecomposable 4-dimensional object
- Just SU(2) matrix approach
- Hopf Coordinates or Two Complex Numbers Approach

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### 3.8 $S^3$ Scalar + Vector Approach, 3D vector and scalar are separate

Thus far, this approach has been unsuccessful in fully resolving the problem. Despite this, we have gained some insight into the problem from it.

The reason that this approach seemed promising was

- Conjugation action is simple and intuitive
- One of the equations we get has a really nice geometric interpretation.
- Vector algebra encodes the SU(2) information nicely and intuitively into 3D rotations and stuff.

Recall that $S^3 \cong \mathrm{SU}(2)$ as Lie Groups. Via this isomorphism, one may show that $\mathrm{tr}(A) = 2 \cdot \mathrm{Re}(A)$, where on the left-hand-side $A \in \mathrm{SU}(2)$ and on right-hand-side is the element of $S^3$ corresponding to $A$. Henceforth, I will use the same symbol interchangeably between the matrix or 4-vector definition. Moreover, a unit length quaternion $A \in S^3$ has a natural interpretation as $A = r + \mathbf{u}$, where $r \in \mathbb{R}$ and $\mathbf{u} \in \mathbb{R}$ such that $r^2 + ||\mathbf{u}|| = 1$.

By the isomorphism above, the vector algebra on these unit length quaternions encode the above determinant 1 condition and the orthonormality of columns conditions. Moreover, there is a nice formula to multiply two elements in $S^3$. If $A = r + \mathbf{u} \in S^3$ and $B = s + \mathbf{v} \in S^3$, then

$$AB = (rs - \mathbf{u} \cdot \mathbf{v}) + (r\mathbf{v} + s\mathbf{u} + \mathbf{u} \times \mathbf{v}). \tag{28}$$

Multiplying out the triple product, one finds that (if we denote $C = t + \mathbf{w} \in S^3$ and use $A, B$ as above).

$$\mathrm{Re}(ABC) = rst - t(\mathbf{u} \cdot \mathbf{v}) - s(\mathbf{u} \cdot \mathbf{w}) - r(\mathbf{v} \cdot \mathbf{w}) - \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}), \tag{29}$$
$$\mathrm{Re}(BAC) = rst - t(\mathbf{u} \cdot \mathbf{v}) - s(\mathbf{u} \cdot \mathbf{w}) - r(\mathbf{v} \cdot \mathbf{w}) + \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}). \tag{30}$$

One finds that the conditions of $\mathrm{tr}(ABC) = \mathrm{tr}(BAC) = 0$ are equivalent those of $\mathrm{tr}(ABC) + \mathrm{tr}(BAC) = 0$ and $\mathrm{tr}(ABC) - \mathrm{tr}(BAC) = 0$. Substituting in equations (29) and (30), one finds that these conditions are equivalent to those of

$$0 = \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}), \tag{31}$$
$$0 = rst - t(\mathbf{u} \cdot \mathbf{v}) - s(\mathbf{u} \cdot \mathbf{w}) - r(\mathbf{v} \cdot \mathbf{w}). \tag{32}$$

We have shown that

$$\begin{aligned}\mathcal{R}(T^2, 2) \cong \Big\{ (r + \mathbf{u}, s + \mathbf{v}, t + \mathbf{w}) \in (S^3)^3 : r^2 + ||\mathbf{u}||^2 &= 1, \\ s^2 + ||\mathbf{v}||^2 &= 1, \\ t^2 + ||\mathbf{w}||^2 &= 1, \\ \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) &= 0, \\ rst - t(\mathbf{u} \cdot \mathbf{v}) - s(\mathbf{u} \cdot \mathbf{w}) - r(\mathbf{v} \cdot \mathbf{w}) &= 0 \Big\}.\end{aligned} \tag{33}$$

The problem now reduces to understanding what this set is.

One approach we took is to fully commit to the 3D vector approach. This requires rewriting $r = \pm\sqrt{1 - ||\mathbf{u}||^2}$, etc. in the last equation above. We then get a map $\mathcal{R}(T^2, 2) \to \overline{B_0(1)}^3$, where $\overline{B_0(1)} := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1\}$ given by $(r + \mathbf{u}, s + \mathbf{v}, t + \mathbf{w}) \mapsto (\mathbf{u}, \mathbf{v}, \mathbf{w})$. The image of this map is

$$\begin{aligned}&\{(\mathbf{u}, \mathbf{v}, \mathbf{w}) \in \overline{B_0(1)}^3 : \\ &\quad \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) = 0, \\ &\quad (\pm_u 1)\sqrt{(1 - ||\mathbf{u}||^2}(\pm_v 1)\sqrt{1 - ||\mathbf{v}||^2}(\pm_w 1)\sqrt{1 - ||\mathbf{w}||^2)} \mp_w \sqrt{1 - ||\mathbf{w}||^2}(\mathbf{u} \cdot \mathbf{v}) \mp_v \sqrt{1 - ||\mathbf{v}||^2}(\mathbf{u} \cdot \mathbf{w}) \mp_u \sqrt{1 - ||\mathbf{u}||^2}(\mathbf{v} \cdot \mathbf{w}) = 0\},\end{aligned} \tag{34}$$

where we have eight choices of $(\pm_u, \pm_v, \pm w)$ (i.e. the above set has these eight equations connected with logical OR).

We took four approaches to investigate the sets identified in equation (33) and (34). We cover them in the following four subsections.

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#### 3.8.1 Gluing Approach

**Claim: There is a bijective correspondence** $H := \{(c_2, u_1, u_2) \in \mathbb{R} \times (\mathbb{R}^3)^2 : \exists! u_3 \in$ **span**$\langle u_1, u_2 \rangle$ **s.t.** $u_1 \cdot u_3 = c_2, u_2 \cdot u_3 = 1 - u_1 \cdot u_2 - c_2\} \cong \{(a, b, c) \in (\mathbb{R}^3)^3 : a, b, c, 0$ **are coplanar**, $1 = a \cdot b + a \cdot c + b \cdot c\}$ **given by the map** $(c_2, u_1, u_2) \mapsto (u_1, u_2, u_3)$**.** Use the moment problem to show the bijection.

**Claim:** $H = \mathbb{R} \times (\mathbb{R}^3)^2$**.** To show this, use the moment problem.

Therefore, we get an 8-to-1 covering map

$$\mathcal{R}(T^2, 2) \cap \{r \neq 0, s \neq 0, t \neq 0\} \to \{(a, b, c) \in (\mathbb{R}^3)^3 : a, b, c, 0 \text{ are coplanar}, 1 = a \cdot b + a \cdot c + b \cdot c\}$$

given by

$$(r + \mathbf{u}, s + \mathbf{v}, t + \mathbf{w}) \mapsto \left(\frac{||\mathbf{u}||}{r}\hat{\mathbf{u}}, \frac{||\mathbf{v}||}{s}\hat{\mathbf{v}}, \frac{||\mathbf{w}||}{t}\hat{\mathbf{w}}\right). \tag{35}$$

The map is 8-to-1 because $\frac{||-\mathbf{u}||}{-r}(-1)\hat{\mathbf{u}} = \frac{||\mathbf{u}||}{r}\hat{\mathbf{u}}$. For example, $\mathcal{R}(T^2, 2) \cap \{r > 0, s > 0, t > 0\} \cong \mathbb{R} \times (\mathbb{R}^3)^2$, a homeomorphism.

To understand the codomain of this map better, we tried analyzing the fibers of the dot product map $f : (\mathbb{R}^3)^2 \to \mathbb{R}$ given by $(a, b) \mapsto a \cdot b$. Specifically we sought a free and transitive group action that preserved each fiber. We failed in doing so because we cannot get a free group action (i.e. one without the $S^1$ roll action as found in the Hopf fibration) that obtains all the rotational degrees of freedom, for that requires a (probably) smooth group structure on $S^2$, which does not exist. Moreover, we did not succeed in finding a transitive group action that preserved the fibers because our attempt consisted of combining an SO(3) diagonal action with a multiplication action $\mathbb{R}^*$ acts on $(\mathbb{R}^3)^2$ via $h, a, b \mapsto (ha, \frac{1}{h}b)$. Combining these won't get every fiber element because we miss those vectors which have a larger relative angle and long enough so as to preserve the dot product.

We computed the fibers of the dot product a different way (**We're not exactly sure how this fits in with this section, but we put it here anyway.**). Fix $a \in \mathbb{R}^3$ and $d \in \mathbb{R}$, now find all $b$ such that $f(a, b) = d$. Let $a \neq 0$. If $d = 0$, then $b = 0$ OR $b \perp a$, so the solution space of $b$ is $\mathbb{R}^2$ (fiber). If $d \neq 0$, then $b \neq 0$ AND $\cos(\theta) \neq 0$, where $\theta$ is relative angle. Then $||b|| = \frac{d}{||a|| \cos(\theta)}$ with case i) $d > 0$ and $\cos(\theta) > 0$ OR case ii) $d < 0$ AND $\cos(\theta) < 0$. In case i) given $\theta \in [0, 90°)$ which describes orientation in a plane, $\exists! b$ in that plane with $f(a, b) = d$. Then we may do a rotation about $a$ to get all other elements in the fiber. In case 2) given $\theta \in (90°, 180°]$, $\exists! b$ in a plane with that orientation. Similarly rotate around $a$ to get all elements in the fiber. Combining the cases, we get a disconnected hyperbolic shape with the property that the union of the fibers over $d \in \mathbb{R}$ is $\mathbb{R}^3$. If $a = 0$, then the fiber is $\emptyset$ if $d \neq 0$ and $\mathbb{R}^3$ else. Furthermore, if $a \neq 0$ and $d$ limits to 0, then the hyperbolic shape limits to $\mathbb{R}^2$.

Boundary Cases.

1. $\mathcal{R}(T^2, 2) \cap \{r = 0, s = 0, t \neq 0\}$. This is equal to

$$\{(\hat{\mathbf{u}}, s + \mathbf{v}, t + \mathbf{w}) \in S^2 \times (S^3)^2 : \hat{\mathbf{u}}, \mathbf{v}, \mathbf{w}, \mathbf{0} \text{ are coplanar}, t(\hat{\mathbf{u}} \cdot \mathbf{v}) + s(\hat{\mathbf{u}} \cdot \mathbf{w}) = 0, s^2 + \mathbf{v}^2 = t^2 + \mathbf{w}^2 = 1, s \neq 0, t \neq 0\}.$$

For each $\hat{\mathbf{u}}$, map this set onto $\{(a,b) \in (\mathbb{R}^3)^2 : a+b \perp \hat{\mathbf{u}}\} \cong \mathbb{R}^2$ via $(\hat{\mathbf{u}}, s+\mathbf{v}, t+\mathbf{w}) \mapsto \left(\frac{\mathbf{v}}{s}, \frac{\mathbf{w}}{t}\right)$. This is a 4-to-1 covering map with fibers homeomorphic to $\mathbb{R}^2$, total space $\mathcal{R}(T^2,2) \cap \{r=0, s=0, t \neq 0\}$, and base space $S^2$.

2. $\mathcal{R}(T^2,2) \cap \{r=0, s=0, t \neq 0\}$. This set equal to $\{(\hat{\mathbf{u}}, \hat{\mathbf{v}}, t+\mathbf{w}) \in (S^2)^2 \times S^3 : \hat{\mathbf{u}}, \hat{\mathbf{v}}, \mathbf{w}, \mathbf{0}$ are coplanar, $t(\hat{\mathbf{u}} \cdot \hat{\mathbf{v}}) = 0, t^2 + \mathbf{w}^2 = 1, t \neq 0\}$. For each $\hat{\mathbf{u}}, \hat{\mathbf{v}}$ such that $\hat{\mathbf{u}} \cdot \hat{\mathbf{v}} = 0$ (We believe with this relation that $\hat{\mathbf{u}}, \hat{\mathbf{w}}$ come from $S^3$), map this set onto $\mathbb{R}^2$ via $\mathbf{w} \mapsto \mathbf{w}/t$ (This doesn't really make sense, but I don't have time to re-analyze it.) Thus we get a fiber bundle with fibers homeomorphic to $\mathbb{R}^2$ and base space $S^3$.
3. $\mathcal{R}(T^2,2) \cap \{r=0, s=0, t=0\}$. This set is $\{(\hat{\mathbf{u}}, \hat{\mathbf{v}}, \hat{\mathbf{w}}) \in (S^2)^3 : \hat{\mathbf{u}}, \hat{\mathbf{v}}, \hat{\mathbf{w}}, \mathbf{0}$ are coplanar$\}$. We get that this is a fiber bundle with fibers homeomorphic to $S^1$ and base space $S^2 \times S^2$.

Some further comments on this approach.

- We briefly connected this approach to projective geometry and how it might relate to the 8-to-1 cover of $\mathcal{R}(T^2,2)_{\mathrm{SU}(2)} \to \mathcal{R}(T^2,2)_{\mathrm{SO}(3)}$. **What are the fibers of the induced map** $\chi(T^2,2)_{\mathrm{SU}(2)} \to \chi(T^2,2)_{\mathrm{SO}(3)}$? Is it 4-to-1?
- One may relate $a \cdot b + a \cdot c + b \cdot c$ to an energy and momentum interpretation.

---

#### 3.8.2 Stereographic Projection

Similar to the approach of Section 3.8.1.

Stereographic projection (about the point $(-1,0,0,0)$) is the correspondence of $p = (x_0, x_1, x_2, x_3) \in S^3 \setminus \{(-1,0,0,0)\} \mapsto \mathbf{u} = \frac{1}{1+x_0}(x_1, x_2, x_3) \in \mathbb{R}^3$. The inverse of this map is $\mathbf{u} \in \mathbb{R}^3 \mapsto \left(\frac{1-||\mathbf{u}||^2}{1+||\mathbf{u}||^2}, \frac{2\mathbf{u}}{1+||\mathbf{u}||^2}\right) \in S^3 \setminus \{(-1,0,0,0)\}$. We saw some similarities between this map and the map (3.8.1). Some good things about stereographic projection is that the vector part of $p$ is in the span of the image $\mathbf{u}$ of $p$, which implies that equation 31) is preserved. Furthermore, the square roots are removed from equation (32). Specifically, if

$$r = \frac{1-||\mathbf{z}_1||^2}{1+||\mathbf{z}_1||^2}, \mathbf{u} = \frac{2\mathbf{z}_1}{1+||\mathbf{z}_1||^2},$$

$$s = \frac{1-||\mathbf{z}_2||^2}{1+||\mathbf{z}_2||^2}, \mathbf{v} = \frac{2\mathbf{z}_2}{1+||\mathbf{z}_2||^2},$$

$$t = \frac{1-||\mathbf{z}_3||^2}{1+||\mathbf{z}_3||^2}, \mathbf{w} = \frac{2\mathbf{z}_3}{1+||\mathbf{z}_1||^3},$$

then equation (32) is

$$(1-||\mathbf{z}_1||^2)(1-||\mathbf{z}_2||^2)(1-||\mathbf{z}_3||^2) - (1-||\mathbf{z}_3||^2)4\mathbf{z}_1 \cdot \mathbf{z}_2 - (1-||\mathbf{z}_2||^2)4\mathbf{z}_1 \cdot \mathbf{z}_3 - (1-||\mathbf{z}_1||^2)4\mathbf{z}_2 \cdot \mathbf{z}_3 = 0. \quad (36)$$

**This is as far as our analysis went on this approach. Perhaps more work is to be done now that square roots are absent.**

---

#### 3.8.3 Angular Substitution Attempt

Start with equation (32). Make the substitutions (invertible coordinate transformation)

$$r = \cos(\alpha), \mathbf{u} = \sin(\alpha) I_\alpha,$$

$$s = \cos(\beta), \mathbf{v} = \sin(\beta) I_\beta,$$

$$t = \cos(\gamma), \mathbf{w} = \sin(\gamma) I_\gamma,$$

for $\alpha, \beta, \gamma \in (0, \pi)$ and $I_\theta$ is the vector part of the quaternion divided by $\sin(\theta)$. Now by enforcing the coplanarity of $I_\alpha, I_\beta, I_\gamma, 0$ along with finding a unique representative of each conjugacy class, we may write $I_\alpha = i, I_\beta = \cos(\xi)i + \sin(\xi)j, I_\gamma = \cos(\eta)i + \sin(\eta)j$ with $(\xi, \eta) \sim (-\xi, -\eta)$. Plugging everything into equation (32), we find that this equation is equivalent to the following information:

$$\cos(\alpha)\cos(\beta)\cos(\gamma) - \cos(\gamma)\sin(\alpha)\sin(\beta)\cos(\xi) - \cos(\beta)\sin(\alpha)\sin(\gamma)\cos(\eta) - \cos(\alpha)\sin(\beta)\sin(\gamma)\cos(\xi - \eta) = 0 \quad (37)$$

with additional data of $\alpha, \beta, \gamma \in (0, \pi)$ and $(\xi, \eta) \sim (-\xi, -\eta)$.

This approach does not work because we are taking a representative of the conjugacy class first and then plugging that into the remaining equation, which is an artificial placement on the set. We prefer to find a natural parameterization of the equation and then get a unique representative from that.

The remaining equation is also abstruse. We also get a bunch of equivalence relations which makes it even more difficult see what's going on.

**Q: How does this relate to spherical geometry? Is the denominator of $\sqrt{1 - x^2}$ show up in spherical geometry?**

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### 3.8.4 Hyperbolic Trigonometry Approach

See [6] for information regarding hyperbolic trigonometry.

There is a map $S^3 \to \mathbb{D}^3 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^1 \le 1\}$ given by $r + \mathbf{u} \mapsto \mathbf{u}$. Apply this map to $\mathcal{R}(T^2, 2)$ to get three points in $\mathbb{D}^3$ coplanar with $\mathbf{0}$. We may restrict $\mathbb{D}^3$ and its hyperbolic metric to that disc, which becomes a Poincare 2-disc with metric $d(\mathbf{u}, \mathbf{v}) = 2\sinh^{-1}\left(\frac{||\mathbf{u}-\mathbf{v}||}{\sqrt{(1-||\mathbf{u}||^2)(1-||\mathbf{v}||^2)}}\right)$, where $||\cdot||$ is the Euclidean norm. Denote $d_u = d(\mathbf{u}, 0), d_v = d(\mathbf{v}, 0), d_w = d(\mathbf{w}, 0)$. Take equation (34), divide it by $rst$ ($r \ne 0, s \ne 0, t \ne 0$), and introduce the formula $\sinh\left(\frac{d(\mathbf{u},0)}{2}\right) = \frac{||\mathbf{u}||}{\sqrt{1-||\mathbf{u}||^2}}$ to get

$$\begin{aligned}1 \\ =&(\pm_u 1)(\pm_v 1)\sinh\left(\frac{d_u}{2}\right)\sinh\left(\frac{d_v}{2}\right)\cos(\theta_{uv}) + (\pm_u 1)(\pm_w 1)\sinh\left(\frac{d_u}{2}\right)\sinh\left(\frac{d_w}{2}\right)\cos(\theta_{uw}) + (\pm_v 1)(\pm_w 1)\sinh\left(\frac{d_v}{2}\right)\sinh\left(\frac{d_w)}{2}\right)\cos(\theta_{vw})\end{aligned} \quad (38)$$

Use the hyperbolic law of cosines $\cosh(c) = \cosh(a)\cosh(b) - \sinh(a)\sinh(b)\cos(\gamma)$ on the inner three sub-triangles (one may check that these are well-defined) of Figure 2. Depending on what $(\pm_u, \pm_v, \pm_w)$ is, we can add/subtract these three hyperbolic law of cosines in the corresponding way to get a relation **independent of the inner $\theta$ angles** (which is thus a statement about the intermediate triangle). For example, if we take $(\pm_u, \pm_v, \pm_w) = (+, +, +)$ and multiply each law of cosines by $-1$ and then add all of them, we get the relation (referring to Figure 2)

$$1 = \cosh\left(\frac{d_v}{2}\right)\cosh\left(\frac{d_w}{2}\right) + \cosh\left(\frac{d_w}{2}\right)\cosh\left(\frac{d_u}{2}\right) + \cosh\left(\frac{d_u}{2}\right)\cosh\left(\frac{d_v}{2}\right) - \cosh\left(d'_{vw}\right) - \cosh\left(d_{uw'}\right) - \cosh\left(d'_{uv}\right). \quad (39)$$

Some thoughts we've been having are as follows. We want to define a measurement of a hyperbolic triangle that is finite even for ideal triangles. This naturally leads to area and angles. We are unsure of the connection between equation (38) and areas/angles. Perhaps there is a better trigonometric view of this equation. **The condition of $\Delta = 1 - \cosh^2(a) - \cosh^2(b) - \cosh^2(c) + 2\cosh(a)\cosh(b)\cosh(c) > 0$ for side lengths $a, b, c$ to form a hyperbolic triangle (this inequality breaks down for ideal triangles... What's going on?) is similar to the inequality mentioned in Section 3.5.**

At this stage it is not clear how to proceed with this approach. Some drawbacks of this approach are that the map is 8-to-1, and the map depends where the origin is. This second point means it is not truly a hyperbolic gadget because the triangle is invariant under the Mobius transformations that fix the origin. Maybe there is a better way to define the hyperbolic map, but if one cannot fix the 8-to-1 issue, it's not that worthwhile. It is also not clear how the residual axis rotation conjugation factors in nicely with the hyperbolic structure.

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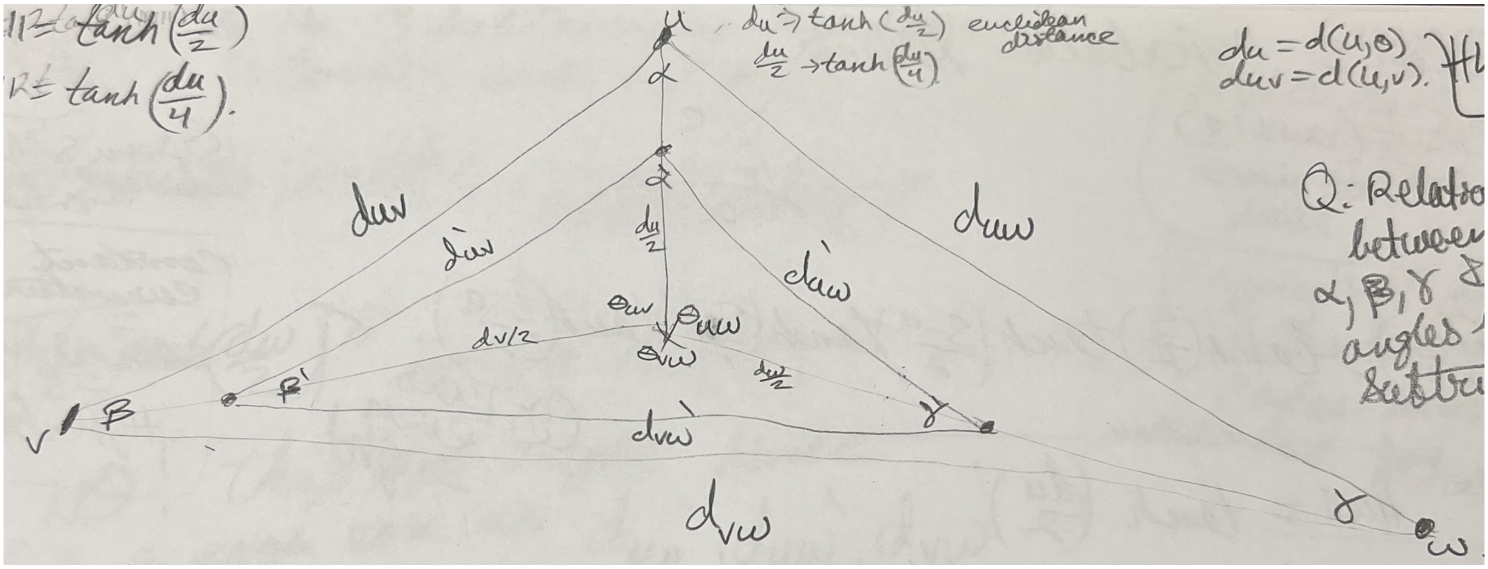


Figure 2: Triangle related to certain $A, B, C \in \mathcal{R}(T^2, 2)$ under the map $S^3 \to \mathbb{D}^3$.

### 3.8.5 Interesting Dot Product Formulation

SEE PAGE H30.5 for more information. Equation (32) may be written as

$$rst = \begin{pmatrix} r \\ s \\ t \end{pmatrix} \cdot \begin{pmatrix} \mathbf{v}\cdot\mathbf{w} \\ \mathbf{u}\cdot\mathbf{w} \\ \mathbf{u}\cdot\mathbf{v} \end{pmatrix}. \tag{40}$$

This implies the equation

$$\frac{r^2 s^2 t^2}{r^2+s^2+t^2} = [(\mathbf{v}\cdot\mathbf{w})^2 + (\mathbf{u}\cdot\mathbf{w})^2 + (\mathbf{u}\cdot\mathbf{v})^2]\cos^2(\theta), \tag{41}$$

where $\theta$ is the angle between these two abstract vectors. We do not know what to make of this construction.

---

### 3.8.6 Interesting Identity

For $A = r + \mathbf{u}, B = s + \mathbf{v}, C = t + \mathbf{w} \in S^3$, the information of $[\mathbf{w}\cdot(\mathbf{u}\times\mathbf{v})]^2 = 0$ and $r^2 + \mathbf{u}^2 = s^2 + \mathbf{v}^2 = t^2 + \mathbf{w}^2 = 1$ implies (but does not seem equivalent to) the following equation

$$\begin{aligned} &0 \\ =&[1-\frac{1}{4}\mathrm{tr}^2(A)][1-\frac{1}{4}\mathrm{tr}^2(B)][1-\frac{1}{4}\mathrm{tr}^2(B)] - [1-\frac{1}{4}\mathrm{tr}^2(C)][\frac{1}{4}\mathrm{tr}(A)\mathrm{tr}(B) - \frac{1}{2}\mathrm{tr}(AB)] \\ &- [1-\frac{1}{4}\mathrm{tr}^2(A)][\frac{1}{4}\mathrm{tr}(B)\mathrm{tr}(C) - \frac{1}{2}\mathrm{tr}(BC)] - [1-\frac{1}{4}\mathrm{tr}^2(B)][\frac{1}{4}\mathrm{tr}(A)\mathrm{tr}(C) - \frac{1}{2}\mathrm{tr}(AC)] \\ &+ 2[\frac{1}{4}\mathrm{tr}(A)\mathrm{tr}(B) - \frac{1}{2}\mathrm{tr}(AB)][\frac{1}{4}\mathrm{tr}(A)\mathrm{tr}(C) - \frac{1}{2}\mathrm{tr}(AC)][\frac{1}{4}\mathrm{tr}(B)\mathrm{tr}(C) - \frac{1}{2}\mathrm{tr}(BC)]. \end{aligned} \tag{42}$$

One receives this by expanding out $[\mathbf{w}\cdot(\mathbf{u}\times\mathbf{v})]^2 = 0$ using vector algebra identities. **We have not thought any further about this equation, although there does seem to be similarities with equation (32).**

---

### 3.9 A Substitution

Take the equations in the set (24) and introduce the variables $u_1, u_2, u_3, v_1, v_2, v_3 \in \mathbb{R}$ via the coordinate transformation

$$t_1 = u_1 + v_1, \tag{43}$$
$$t_{23} = u_1 - v_1, \tag{44}$$
$$t_2 = u_2 + v_2, \tag{45}$$
$$t_{13} = u_2 - v_2, \tag{46}$$
$$t_3 = u_3 + v_3, \tag{47}$$
$$t_{12} = u_3 - v_3. \tag{48}$$

Note that $P(T) = Q(T) = 0 \Leftrightarrow \frac{1}{2}(P(T) + \frac{1}{2}Q(T)) = \frac{1}{2}(P(T) - \frac{1}{2}Q(T))$. Using this new form and the variable substitution, the system is equivalent to

$$1 = u_1^2 + u_2^2 + u_3^2 - u_1u_2u_3 - u_1u_2v_3 - u_1v_2u_3 - v_1u_2u_3, \tag{49}$$
$$1 = v_1^2 + v_2^2 + v_3^2 + v_1v_2v_3 + v_1v_2u_3 + v_1u_2v_3 + u_1v_2v_3. \tag{50}$$

We get six variables and two equations which are dual.

---

#### 3.9.1 A Lemma

This is relevant for the Quadratic Form approach below when we talk about eigenvalue = 0 and singular points (disclaimer: this idea is not fully developed yet).

**Lemma:** Let $A, B, C \in S^3$. Then $Z_A \cap Z_B \cap Z_C \supset \{-1, 0, 1\}$ (proper subset, algebra centralizers) $\Leftrightarrow A, B, C$ all share the same axis or are some/all are purely real.

*Our Solution.* Sublemma: $Z_A \cap S^3 = \{h \in S^3 | h \text{ has same or 0-axis (purely real)}\}$. This sublemma follows by expressing $A$ and $h$ in angle-axis forms and then multiplying out and noting that the cross product of their axes is zero. SUBLEMMA PROVED. By hypothesis of the main lemma, $\exists h \in Z_A \cap Z_B \cap Z_C \setminus \{-1, 0, 1\}$, which is equivalent to $\exists \hat{h} \in Z_A \cap S^3 \cap Z_B \cap S^3 \cap Z_C \cap S^3 \setminus \{-1, 0, 1\}$. By sublemma, $\hat{h}$ is purely real (0-axis) OR it has the same axis as $A, B, C$. $\hat{h} \notin \mathbb{R}$, for then $\hat{h} = \pm 1$, false. Thus, $\hat{h}$ has a non-degenerate axis. Now we have a few cases

1. None of $A, B, C$ is real. Then $\hat{h}$ has the same axis as all three, which implies that $A, B, C$ all have the same axis.
2. Exactly one of $A, B, C$ is purely real. Then $\hat{h}$ has the same axis as the two non-real quaternions.
3. At least two of $A, B, C$ are purely real. This case is trivial since centralizer of real numbers is the whole space.

■

One may apply this classification to the additional conditions within $\mathcal{R}(T^2, 2)$ to get 8 equations (the dot products simplify since the vector parts would be collinear with 0). **We haven't thought too much about this.**

---

#### 3.9.2 Dimensional Reduction

Provide the example calculations I did on those sheets. Notice that we are just diagonalizing and also want the eigenvalues to be WHAT.

Motivates the quadratic form and diagonalization approach.

In equations (49) and (50), put $u_1 = v_3 = 0$. They become

$$1 = u_2^2 + u_3^2 - v_1u_2u_3, \tag{51}$$
$$1 = v_1^2 + v_2^2 + v_1v_2u_3. \tag{52}$$

Make the substitution (valid since $\forall i, u_i, v_i \in [-2, 2]$

$$m_u = \frac{1}{2}\sqrt{2 - v_1}(u_2 + u_3), \tag{53}$$
$$n_u = \frac{1}{2}\sqrt{2 + v_1}(u_2 - u_3), \tag{54}$$
$$m_v = \frac{1}{2}\sqrt{2 + u_3}(v_1 + v_2), \tag{55}$$
$$n_v = \frac{1}{2}\sqrt{2 - u_3}(v_1 - v_2). \tag{56}$$

Then equations (51) and (52) are equivalent to

$$1 = m_u^2 + n_u^2, \tag{57}$$
$$1 = m_v^2 + n_v^2. \tag{58}$$

**Lemma: If we view the $u_i, v_i$ variables as coming from $\mathcal{R}(T^2, 2)$, then $\forall i, u_i, v_i \in (-2, 2)$.**

*Our Solution.* We'll do the example of $v_1$, but by symmetry this argument extends to the others. We have $v_1 = \frac{1}{2}[\text{tr}(A) - \text{tr}(BC)]$ for some $A, B, C \in \text{SU}(2)$ by hypothesis. Write $A = r + \mathbf{u}, B = s + \mathbf{v}, C = t + \mathbf{w}$. If $v_1 = 2$, then $\text{tr}(A) = 2$ and $\text{tr}(BC) = -2$ so that $r = 1, \mathbf{v} = \mathbf{0}, st - \mathbf{v} \cdot \mathbf{w} = -1$. By equation (32), $st - \mathbf{v} \cdot \mathbf{w} = 0$, a contradiction. We get a similar contradiction if $v_1 = -2$. ■

This particular example of $u_1 = v_3 = 0$ generalizes. If $u_i = v_j = 0$ for $i \neq j$, then the substitution is (subscripts wrap around):

$$m_u = \frac{1}{2}\sqrt{2 - v_i}(u_{i+1} + u_{i+2}), \tag{59}$$
$$n_u = \frac{1}{2}\sqrt{2 + v_1}(u_{i+1} - u_{i+2}), \tag{60}$$
$$m_v = \frac{1}{2}\sqrt{2 + u_j}(v_{j+1} + v_{j+2}), \tag{61}$$
$$n_v = \frac{1}{2}\sqrt{2 - u_j}(v_{j+1} - v_{j+2}). \tag{62}$$

If $u_i = v_i = 0$, then we get a decoupled set of circles.

We tried using blind algebraic guesses to find the 6-dimensional generalization, but these did not work. **Our main guess is that diagonalization of the quadratic form is the coordinate transformation we're looking for.**

---

#### 3.9.3 Quadratic Form Approach

The dimensional reduction explorations motivate pursuing quadratic forms and diagonalization.

Write equations (49) and (50) in the following suggestive forms:

$$1 = u_1^2 + u_2^2 + u_3^2 - a_1 u_2 u_3 - a_2 u_1 u_3 - a_3 u_1 u_2 = (u_1 \ u_2 \ u_3) \begin{pmatrix} 1 & -\frac{a_3}{2} & -\frac{a_2}{2} \\ -\frac{a_3}{2} & 1 & -\frac{a_1}{2} \\ -\frac{a_2}{2} & -\frac{a_1}{2} & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \tag{63}$$

$$1 = v_1^2 + v_2^2 + v_3^2 + b_1 v_2 v_3 + b_2 v_1 v_3 + b_3 v_1 v_2 = (v_1 \ v_2 \ v_3) \begin{pmatrix} 1 & \frac{b_3}{2} & \frac{b_2}{2} \\ \frac{b_3}{2} & 1 & \frac{b_1}{2} \\ \frac{b_2}{2} & \frac{b_1}{2} & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}, \tag{64}$$

where $a_1 = b_1 = u_1 + v_1, a_2 = v_2, a_3 = v_3, b_2 = u_2, b_3 = u_3$. Let $A$ be the coefficient matrix consisting of the $a_i$ above, and let $B$ be the coefficient matrix consisting of the $b_i$.

Since $A, B$ are symmetric, by Spectral Theorem we may diagonalize each to orthogonal eigenbases in $\mathbb{R}$. Let $\lambda_{A1}, \lambda_{A2}, \lambda_{A3}$ and $\lambda_{B1}, \lambda_{B2}, \lambda_{B3}$ be the eigenvalues of $A$ and $B$, respectively. Let $\mathbf{v}_{A1}, \mathbf{v}_{A2}, \mathbf{v}_{A3}$ and $\mathbf{v}_{B1}, \mathbf{v}_{B2}, \mathbf{v}_{B3}$ be the corresponding eigenvectors. Let $P_A$ be the matrix whose columns are $\mathbf{v}_{A1}, \mathbf{v}_{A2}, \mathbf{v}_{A3}$, in that order, and let $P_B$ be the corresponding matrix for $\mathbf{v}_{Bi}$. Note tht $P_A$ is a function of $u_1, v_1, v_2, v_3$, and $P_B$ is a function of $v_1, u_1, u_2, u_3$. If

$$\begin{pmatrix} u_1' \\ u_2' \\ u_3' \end{pmatrix} = P_A^{-1} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \tag{65}$$

with $v_i'$ having an analogous definition, then equations (49) and (50) become

$$1 = \lambda_{A1} u_1'^2 + \lambda_{A2} u_2'^2 + \lambda_{A3} u_3'^2, \tag{66}$$

$$1 = \lambda_{B1} v_1'^2 + \lambda_{B2} v_2'^2 + \lambda_{B3} v_3'^2. \tag{67}$$

Both of these equations describe ellipsoids if and only if $\lambda_{Ai}, \lambda_{Bi} \geq 0$ for all $i \in \{1, 2, 3\}$. This is equivalent to $A, B$ both being positive semi-definite.

Sylvester's Criterion states that $A, B$ are positive semidefinite if and only if the following conditions hold:

$$a_2 \leq 0, \tag{68}$$

$$1 - \frac{a_1^2}{4} \geq 0, \tag{69}$$

$$1 - \frac{a_2^2}{4} \geq 0, \tag{70}$$

$$1 - \frac{a_3^2}{4} \geq 0, \tag{71}$$

$$-\frac{a_1}{2} - \frac{a_2 a_3}{4} \geq 0, \tag{72}$$

$$\frac{a_2}{2} + \frac{a_1 a_3}{4} \geq 0, \tag{73}$$

$$-\frac{a_3}{2} - \frac{a_1 a_2}{4} \geq 0, \tag{74}$$

$$1 - \frac{a_1^2}{4} - \frac{a_2^2}{4} - \frac{a_3^2}{4} - \frac{a_1 a_2 a_3}{4} \geq 0, \tag{75}$$

$$b_2 \geq 0, \tag{76}$$

$$1 - \frac{b_1^2}{4} \geq 0, \tag{77}$$

$$1 - \frac{b_2^2}{4} \geq 0, \tag{78}$$

$$1 - \frac{b_3^2}{4} \geq 0, \tag{79}$$

$$\frac{b_1}{2} - \frac{b_2 b_3}{4} \geq 0, \tag{80}$$

$$-\frac{b_2}{2} + \frac{b_1 b_3}{4} \geq 0, \tag{81}$$

$$\frac{b_3}{2} - \frac{b_1 b_2}{4} \geq 0, \tag{82}$$

$$1 - \frac{b_1^2}{4} - \frac{b_2^2}{4} - \frac{b_3^2}{4} - \frac{b_1 b_2 b_3}{4} \geq 0. \tag{83}$$

We have the following outlook of the situation:

$$\chi(T^2, 2) \longrightarrow \{(t_1, t_2, t_3, t_{12}, t_{13}, t_{23}) \in [-2, 2]^6 : P(T) = Q(T) = 0, \text{ extra conditions}\}$$

$$\swarrow$$

$$\{(u_1, u_2, u_3, v_1, v_2, v_3) \in (-2, 2)^6 : 1 = u_1^2 + \dots, 1 = v_1^2 + \dots, \text{ extra conditions}\} \longrightarrow \{(m, n, \ell, m', n', \ell') \in \mathbb{R}^6 : m^2 + n^2 + \ell^2 = 1, m'^2 + n'^2 + \ell'^2 = 1\}$$

That is, we want to identify what the "extra conditions" are on the $(u_1, u_2, ...)$ set in order for the departing arrow to be a well-defined bijection. Then we hope to backtrack that to the "extra conditions" on the $(t_1, ...)$ set so that that incoming arrow is also a surjection. I.e. we hope that the SU(2) conditions we're missing imply that condition. This backtracking idea is possible since we already know that the $(m, n, ...)$ set, our final destination, is supposed to be $S^2 \times S^2$, according to Boozer. Our current guess is that the non-negativity of the eigenvalues $\lambda_{Ai}, \lambda_{Bi} \geq 0$ are these conditions on the $(u, v, ...)$ set.

**Claim: The last arrow in the above diagram, the one between the $u-v$ set and the $m-n$ set ($S^2 \times S^2$) actually exists (well-defined map) IF we enforce $\lambda_{Ai}, \lambda_{Bi} \geq 0$.**

***Our Solution.*** Define the sets $M_1 = \{(u_1, u_2, u_3, v_1, v_2, v_3) : 1 = u_1^2 + u_2^2 + u_3^2 - u_1u_2u_3 - u_1u_2v_3 - u_1v_2u_3 - v_1u_2u_3,\ 1 = v_1^2 + v_2^2 + v_3^2 + v_1v_2v_3 + v_1v_2u_3 + v_1u_2v_3 + u_1v_2v_3, \forall i \lambda_{Ai}, \lambda_{Bi} \geq 0\}$ and $M_2 = \{(u_1', u_2', u_3', v_1', v_2', v_3', \lambda_{A1}, \lambda_{A2}, \lambda_{A3}, \lambda$ $1 = \lambda_{A1}u_1'^2 + \lambda_{A2}u_2'^2 + \lambda_{A3}u_3'^2,\ 1 = \lambda_{B1}v_1'^2 + \lambda_{B2}v_2'^2 + \lambda_{B3}v_3'^2\}$. Define the map $f_{2.1} : M_1 \to M_2$ given by

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ v_1 \\ v_2 \\ v_3 \end{pmatrix} \mapsto \begin{pmatrix} \begin{pmatrix} P_A^{-1} & O \\ O & P_B^{-1} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ v_1 \\ v_2 \\ v_3 \end{pmatrix} \\ \lambda_{A1} \\ \lambda_{A2} \\ \lambda_{A3} \\ \lambda_{B1} \\ \lambda_{B2} \\ \lambda_{B3} \end{pmatrix}.$$

This map is well-defined because $A, B$ are symmetric so all the eigenvalues exist and $A, B$ are diagonalizable. Define the map $f_{2.2} : M_2 \to S^2 \times S^2$ given by

$$\begin{pmatrix} u_1' \\ u_2' \\ u_3' \\ v_1' \\ v_2' \\ v_3' \\ \lambda_{A1} \\ \lambda_{A2} \\ \lambda_{A3} \\ \lambda_{B1} \\ \lambda_{B2} \\ \lambda_{B3} \end{pmatrix} \mapsto \begin{pmatrix} \sqrt{\lambda_{A1}}u_1' \\ \sqrt{\lambda_{A2}}u_2' \\ \sqrt{\lambda_{A3}}u_3' \\ \sqrt{\lambda_{B1}}v_1' \\ \sqrt{\lambda_{B2}}v_2' \\ \sqrt{\lambda_{B3}}v_3' \end{pmatrix}.$$

The map $f_{2.2}$ is well-defined precisely because $\lambda_{Ai}, \lambda_{Bi} \geq 0$. Thus the composition $f_3 = f_{2.2} \circ f_{2.1} : M_1 \to S^2 \times S^2$ is well-defined. ■

We think we get bijection even if some $\lambda_{Ai}, \lambda_{Bi} = 0$, for these are singular points and we'll work out special cases for them.

We currently think that $f_3$ is a bijection under the non-negativity of eigenvalues condition set. To prove bijection, we should not look at $f_{2.1}, f_{2.2}$ separately.

Sylvester's Criterion give inequality conditions on the $u_i, v_i$ for non-negativity of eigenvalues. Perhaps there is a simpler set of inequalities. Perhaps if the discriminants of the characteristic polynomials of the $A, B$ matrices are

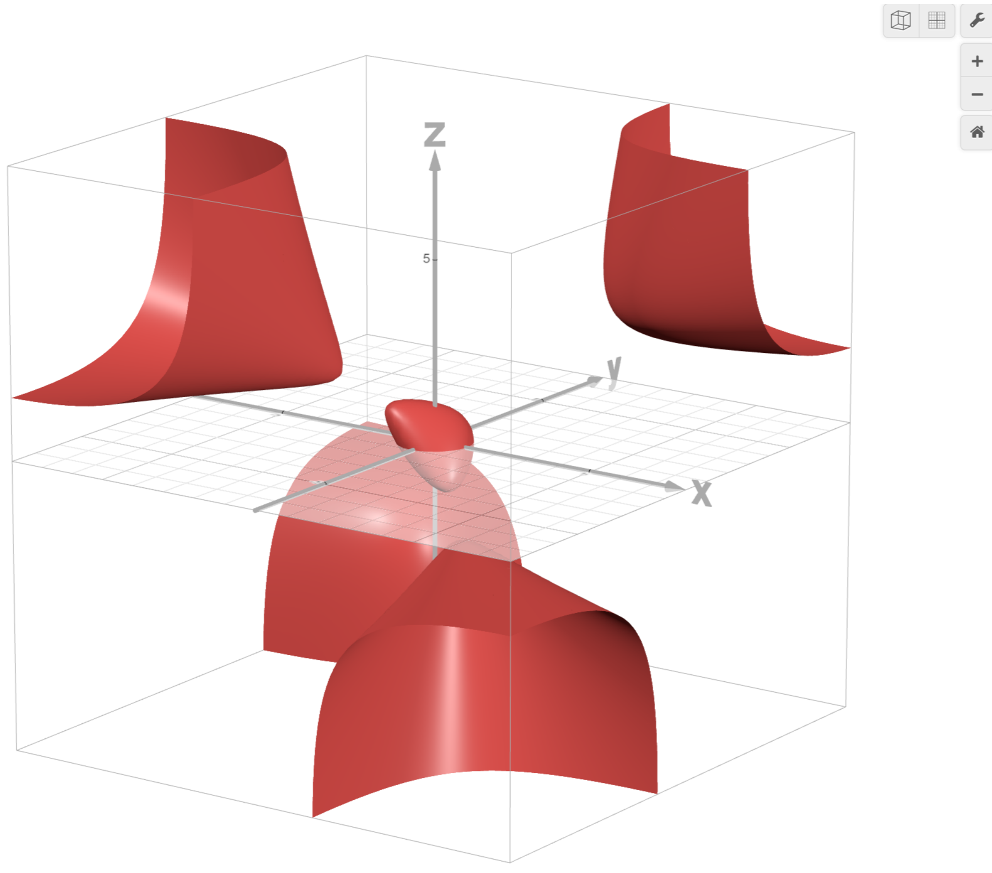


Figure 3: Illustration of equation (49) for $v_1 = 0.12, v_2 = -0.8, v_3 = 0.15$. $u_1, u_2, u_3$ are the $x$-,$y$-, and $z$-axis, respectively.

non-negative and the constant term is non-negative then the roots are real and non-negative and roots $= 0$ correspond to the eigenvalue $= 0$ cases.

The eigenvalue $= 0$ are the singular points of our traceless representation variety we're guessing.

Hard part about all of this: The diagonalization and coordinate transformation depend on $u_1, u_2, u_3, v_1, v_2, v_3$ so it's non-linear.

For an ellipsoid want non-negativity. Technically, we also need it contained in $[-2, 2]^6$ as a necessary condition for trace of SU(2) matrices, but we hope these are implied from the non-negativity of eigenvalues for free.

---

#### 3.9.4 Another Attempt to Enforce Closed Shape

If temporarily define $F(u_1, u_2, u_3; v_1, v_2, v_3) = u_1^2+u_2^2+u_3^2-u_1u_2u_3-u_1u_2v_3-u_1v_2u_3-v_1u_2u_3-1$ and $G(v_1, v_2, v_3; u_1, u_2, u_3) = v_1^2 + v_2^2 + v_3^2 + v_1v_2v_3 + v_1v_2u_3 + v_1u_2v_3 + u_1v_2v_3 - 1$, then equations (49) and (50) become $F(u_1, u_2, u_3, v_1, v_2, v_3) = G(u_1, u_2, u_3, v_1, v_2, v_3) = 0$. We want there to be isolated "inner blob" shapes in the $F = 0$ equation and the $G = 0$ equation. We believe there to be conditions on $v_1, v_2, v_3$ (resp. $u_1, u_2, u_3$) that give the "inner blob" shape for the $F = 0$ (resp. $G = 0$) equation. A necessary condition for the "inner blob" shape to be "closed" or "spheroid" is that the normalized normal vector to the zero set $F = 0$, viewing $v_1, v_2, v_3$ as fixed parameters, on $[-2, 2]^6$ is surjective onto $S^2$ (see Figure 3). The precise question for the $F = 0$ equation is do there exist conditions on $v_1, v_2, v_3$ such that $\forall\theta \in [0, \pi], \forall\phi \in [0, 2\pi], \exists u_1, u_2, u_3 \in [-2, 2]$ satisfying $\frac{\nabla F}{||\nabla F||} = \sin(\theta)\cos(\phi)\hat{\imath} + \sin(\theta)\sin(\phi)\hat{\jmath} + \cos(\theta)\hat{k}$ and $\forall u_1, u_2, u_3, ||\nabla F|| \neq 0$ given the conditions? We proceeded by calculating the gradient of $F$ and then normalizing it. We obtained equations to satisfy, but these were too abstruse to analyze and we abandoned the method. Perhaps there is a better way to portray surjectivity onto $S^2$.

### 3.9.5 Solve System by Symmetry

See [3] for a motivation of the approach. The action by a matrix $M$ on equations (49) and (50) is the variable substitution $(u_1, u_2, u_3, v_1, v_2, v_3) \mapsto M(u_1, u_2, u_3, v_1, v_2, v_3)^t$. The equations are invariant under action by the following classes of matrices:

Type A: $P \otimes I_2$ (Kronecker product I think the order here is backward that found in the Wikipedia page), where $P$ is any $3 \times 3$ permutation matrix, and $I_2$ is the $2 \times 2$ identity matrix and type B2:

$$I_3 \otimes \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and type B1 the block matrix obtained by

$$I_3 \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and multiplying exactly one of the first three rows of the matrix by $-1$. We have not verified that these are all the permutation symmetries of the equations.

The task would now be to identify a simple family of primary invariants for these matrices. We could not find any families that simplified the equations (e.g. coupled symmetric polynomials in u,v). Also the square root in the dimensional reduction had us abandon this approach. **Perhaps there is some approach by "symmetrizing" the equations and then using the elementary symmetric polynomials. We have not looked into this.**

## 3.10 Other Notes and Miscellaneous Approaches

We took brief notes on fiber bundles, principal bundles, hopf fibration, hopf coordinates.

Hopf action is probably unrelated to our conjugation action.

We still haven't fully fleshed out Hopf coordinates approach to this problem

We tried to blindly manipulate the above equations, tried to get squares, tried to get rid of cubics, tried to separate variables to no avail. **Question from Anunoy: Can we even expect the equations to directly give $S^2 \times S^2$ when a lot of the time these character varieties are homeomorphic to pointy pillowcase spheres?** Partial answer: Yes, our $u, v$ equations will have singular points, which correspond to the pillowcase points.

We tried using the conjugation to fix a representative and plug this into the equations (BAD).

Lessons from Boozer's methods: Need to get a unique representative which give you equivalence relations. Need to give a natural global parameterization of the restricting equations THEN enforce the representative from the conjugacy class.

We learned about how free and transitive group actions on a space give homeomorphism. This is a nice way to see the global structure.

We've been trying to fix a few variables and look at the resulting "pseudo-bundles" (not all fibers are homeomorphic), but it is hard to glue these together. Do these even have a name? Everything but a set of measure zero has the same fiber?

We were also trying to look at low dimensional cross sections of the object, but these are also hard to glue together.

We tried enforcing artificial conditions on the equations that are unnatural for their structure.

## 3.11 Future Attempts

Keep trying to flesh out the substitution approach with eigenvalues. Check if enforcing the non-negativity of the eigenvalues indeed gives us a bijection.

Use Hopf (z + wj) coordinates. Treats the 4 components of each quaternion more symmetrically, instead of having 1 real part be the focus. It seems like interpreting a quaternion as a separate real number with a 3D vector is the wrong approach since these quantities are too closely tied together. Try to understand how conjugation works with Hopf coordinates

**Perhaps there is no conceptual leap we are missing. Perhaps it really just an abstruse algebra problem.**

Explore determinantal identities. We noted that equations (49) may be written as

$$
\begin{aligned}
&1 &(84)\\
&= \det\begin{pmatrix}1 & u_1 & u_2\\ u_1 & 1 & u_3\\ u_2 & u_3 & 1\end{pmatrix} - 1 - \frac{1}{2}\left[\det\begin{pmatrix}1 & u_1 & u_2\\ -u_1 & 1 & u_3\\ -u_2 & -u_2 & 1\end{pmatrix} + \det\begin{pmatrix}1 & u_1 & u_2\\ u_1 & 1 & u_3\\ u_2 & u_3 & 1\end{pmatrix}\right]\\
&- \frac{1}{2}\left[\det(123) + \det(-123) + \det(213) + \det(-213) + \det(132) + \det(-132) - 6\right], &(85)
\end{aligned}
$$

where for example the notation $\det(123), \det(-123)$ means

$$
\det(123) = \det\begin{pmatrix}1 & u_1 & u_2\\ v_1 & 1 & u_3\\ v_2 & v_3 & 1\end{pmatrix}, \det(-123) = \det\begin{pmatrix}1 & -u_1 & -u_2\\ v_1 & 1 & -u_3\\ v_2 & v_3 & 1\end{pmatrix}.
$$

For example, $\det(132)$ is obtained by computing $\det(123)$ but swapping the indices on the $u_i, v_i$ as $2 \leftrightarrow 3$. Equation (50) may be expressed similarly.

We did not further study these determinantal identities. Perhaps there is a more succint way of expressing the equations (maybe 4-by-4 matrices?) **We have noticed a relation between the form of the matrices appearing here and those of the proof used in Goldman's paper, mentioned in Section 3.5 of this technical report.** Furthermore, perhaps these determinant equations these are related to the Cayley-Menger determinants, which is related to a generalization of Heron's formula and related to Shoelace formula of expressing area as the sum of determinants. This also somewhat relates to using the area of a hyperbolic triangle to study our problem.

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# Connected components of representation varieties of RAAGs

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## Contents



# 1 Preliminaries

## 1.1 Markings

**Definition 1.1.** An *edge marking* of a graph $(V, E)$ is some $y \in \pi_1(G)^E$ satisfying $y_{(v,v)} = 1$ for any $v \in V$. A *marked graph* is a graph $K$ equipped with a marking $y \in \pi_1(G)^E$.

**Definition 1.2.** The *pullback* of a marking $y \in \pi_1(G)^E$ under a graph homomorphism $f : K' \to K$ is

$$(f^*y)_{(v',w')} = y_{(f(v'),f(w'))}$$

**Definition 1.3.** A *reduction* of a marking $y \in \pi_1(G)^E$ along a graph homomorphism $q : K \to K''$ is some $y'' \in \pi_1(G)^{E''}$ such that $q^*y = y'$.

**Proposition 1.4.** Let $q : K \to K''$ be a graph homomorphism, surjective on vertices and edges. Then $q^*$ is injective on markings.

*Proof.* $q^*$ on markings is a restriction of the map

$$\pi_1(G)^E \xrightarrow{q^*} \pi_1(G)^{E''}$$

Since $E \to E''$ is a surjection and surjections are epimorphisms in **Set**, $q^*$ is injective. □

**Corollary 1.5.** The reduction of a marking along a surjective graph homomorphism is unique, if it exists.

## 1.2 The Second Obstruction Map

**Definition 1.6.** Let $G$ be a Lie group, $K = (V, E)$ be a graph. The *associated commutator map* is

$$\mu_{K,G} : G^V \to G^E \qquad (x_v)_{v\in V} \mapsto ([x_v, x_w])_{(v,w)\in E}$$

**Remark 1.7.** Note that $\mathrm{Hom}(\Gamma_K, G) \cong \mu_{K,G}^{-1}(1)$.

Consider the following diagram

$$\begin{array}{ccc} \widetilde{G}^V & \xrightarrow{\mu_{K,\widetilde{G}}} & \widetilde{G}^E \\ \downarrow & \overset{\overline{\mu}}{\nearrow} & \downarrow \\ G^V & \xrightarrow{\mu_{K,G}} & G^E \end{array}$$

where $\overline{\mu}$ is defined by $x \mapsto \mu_{K,\widetilde{G}}(\widetilde{x})$ where $\widetilde{x}$ is a lift of $x$.

**Lemma 1.8.** $\overline{\mu}(x)$ is independent of choice of lift.

*Proof.* Let $\widetilde{x}$, $\widetilde{x}'$ be distinct lifts of $x$. Then for all $v$, $\widetilde{x}'_v = \widetilde{x}_v c_v$ for some $c_v \in \pi_1(G)$. Let $e = (w, z)$ be an arbitrary edge in $E$. Then since $\pi_1(G) \subseteq Z(\widetilde{G})$,

$$\begin{aligned} \mu_{\pi,\widetilde{G}}(\widetilde{x}')_e &= [\widetilde{x}'_w, \widetilde{x}'_z] = [\widetilde{x}_w c_w, \widetilde{x}_z c_z] \\ &= \widetilde{x}_w c_w \widetilde{x}_z c_z c_w^{-1} \widetilde{x}_w^{-1} c_z^{-1} \widetilde{x}_z^{-1} \\ &= c_w c_w^{-1} c_z c_z^{-1} [\widetilde{x}_w, \widetilde{x}_z] \\ &= [\widetilde{x}_w, \widetilde{x}_z] = \mu_{\pi,\widetilde{G}}(\widetilde{x})_e \end{aligned}$$

□

**Lemma 1.9.** $\overline{\mu}$ is continuous.

*Proof.* Consider an evenly covered neighborhood $U$, so that there is a continuous section $s : U \to \widetilde{G}^V$. Then $\overline{\mu}|_U$ is the composition $\mu_{\widetilde{G}} \circ s$, so $\overline{\mu}|_U$ is continuous. Since the evenly covered neighborhoods of $G^V$ cover $G^V$, $\overline{\mu}$ is continuous. □

**Definition 1.10.** Let $\Gamma_K$ be the RAAG associated to $K$, $G$ be a topological group. The *obstruction map* is the continous map

$$o_K : R(\pi, G) \to \pi_1(G)^E \qquad\qquad x \mapsto \overline{\mu}(x)$$

*Proof of well-definedness.* Since $o(x)$ maps to the identity in $G^E$, by exactness of $\pi_1(G) \to \widetilde{G} \to G$, $o(x)$ lifts to an element of $\pi_1(G)$. □

**Lemma 1.11.** Let $c \in \pi_1(G)$. The natural map $\widetilde{G}^V \to G^V$ restricts to a surjective map $q : \mu^{-1}_{K,\widetilde{G}}(c) \to o_K^{-1}(c)$.

*Proof.* Let $\widetilde{x} \in \mu^{-1}_{K,G}(c)$, $x$ be the image of $\widetilde{x}$ in $G^V$. Since $k$ mapsto the identity in $G^E$, by commutativity of the diagram $\mu_{K,G}(x) = 1$ and thus $x \in \mathrm{Hom}(\Gamma_K, G)$. By construction, $\mu_{\pi,\widetilde{G}}(\widetilde{x}) = \overline{\mu}(x) = o(x)$, so $x \in o^{-1}(c)$.

For surjectivity, any element $x \in o^{-1}(c)$ lifts to some $\widetilde{x} \in \widetilde{G}^V$. Since $\mu_{K,\widetilde{G}}(\widetilde{x}) = o(x)$, we have that $\widetilde{x} \in \mu^{-1}_{\pi,\widetilde{G}}(c)$, as desired. □

**Proposition 1.12** (Naturality)**.** Let $f : K' \to K$ be a graph homomorphism. Then the diagram

$$\begin{array}{ccc} \mathrm{Hom}(\Gamma_K, G) & \xrightarrow{o_K} & \pi_1(G)^E \\ \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} \\ \mathrm{Hom}(\Gamma_{K'}, G) & \xrightarrow{o_{K'}} & \pi_1(G)^{E'} \end{array}$$

commutes; i.e.

$$f^* o_K(x) = o_{K'}(f^* x)$$

*Proof.* Let $(v', w') \in E'$. Then

$$\begin{aligned} (f^* o_K(x))_{(v',w')} &= o_K(x)_{(f(v'),f(w'))} \\ &= [x_{f(v')}, x_{f(w')}] \\ &= [(f^*x)_{v'}, (f^*x)_{w'}] \\ &= o_{K'}(f^*x) \end{aligned}$$

□

**Corollary 1.13** (Functoriality of $o$)**.** Let $y \in \pi_1(G)^{E'}$ be a marking. Then the map $f^* : \mathrm{Hom}(\Gamma_K, G) \to \mathrm{Hom}(\Gamma_{K'}, G)$ restricts to a map $f^* : o_K^{-1}(y) \to o_{K'}^{-1}(f^* y)$.

# 2 SU($n$) and U($n$)-representations

**Notation.** Let $E_{\lambda,A}$ denote the $\lambda-$eigenspace of $A$.

**Lemma 2.1.** Let $A, B \in \mathrm{GL}_n(k)$ with $A$ diagonalizable. Then $[A, B] = zI$ iff

$$B(E_{\lambda,A}) = E_{z\lambda,A}$$

for all eigenvalues $\lambda$ of $A$.

**Notation.** Define

$$C_G(B, z) := \{A \mid [A, B] = z\}$$

We consider Lie groups satisfying the following property.

**Property A** Let $S \subseteq G$ be a finite set. Then the inclusion $Z(G) \to C_G(S)$ induces a surjection on $\pi_0$.

**Lemma 2.2.** Let $A \in C_G(B)$ with $A$ diagonalizable, and suppose for any eigenvalue $\lambda$ of $A$, $\exists \lambda'$ with

$$E_{\lambda,A} \subseteq E_{\lambda',A'}$$

Then $A' \in C_G(B)$.

*Proof.* Let $\{\lambda_{i_k}\}$ denote the eigenvalues of $A$ with $E_{\lambda_{i_k},A} \subseteq E_{\lambda'_i,A'}$, so that $E_{\lambda'_i,A'} \supseteq \bigoplus E_{\lambda_{i_k},A}$. But

$$\bigoplus_i \bigoplus_k E_{\lambda_{i_k},A} = V = \bigoplus_i E_{\lambda'_i,A'}$$

FTSOC suppose $E_{\lambda'_i,A'} \supsetneq \bigoplus E_{\lambda_{i_k},A}$. Then the dimension of the left side is less than the dimension of the right side, contradiction. So $E_{\lambda'_i,A'} = \bigoplus E_{\lambda_{i_k},A}$, and thus $B(E_{\lambda'_i,A'}) = E_{\lambda'_i,A'}$. ☐

**Lemma 2.3.** SU($n$) and U($n$) satisfy property A.

*Proof.* Let $A \in C_G(S)$. Then for some unitary $P$, $A = P \operatorname{diag}\{\lambda_1, ..., \lambda_1, ..., \lambda_r\}P^{-1}$ where $\lambda_i$ appears $e_i$ times. For $r = 1$, $A = \lambda_1 I \in Z(G)$, so suppose $r > 1$. Define

$$Z := \begin{cases} (S^1)^r & G = \mathrm{U}(n) \\ \{(\lambda_1, ..., \lambda_r) \in (S^1)^r \mid \prod \lambda_i^{e_i} = 1\} & G = \mathrm{SU}(n) \end{cases}$$

Note that $Z$ is connected. (todo)

Consider a path $\gamma : [0, 1] \to Z$ from $(\lambda_1, ..., \lambda_r)$ to $(1, ..., 1)$. Then consider the path

$$A(t) = P \operatorname{diag}\{\gamma(t)_1 : e_1, ..., \gamma(t)_r : e_r\}P^{-1}$$

By the previous lemma, $A(t) \in C_G(S)$ and $A(1) \in Z(G)$, as desired. ☐

**Theorem 2.4.** Let $G$ be a connected Lie group satisfying property A. Then $\mathrm{Hom}(\Gamma_K, G)$ is connected.

*Proof.* WLOG suppose $V = \{1, ..., n\}$. Let $x = (x_i)_{i \in V} \in \mathrm{Hom}(\Gamma_K, G)$. We show inductively that $x$ has a path to some $y \in Z(G)^r \times G^{n-r}$.

Suppose $(x_i)$ has a path to some $(y_i) \in Z(G)^{r-1} \times G^{n-r+1}$. Let $S_r$ denote the set of vertices adjacent to $r$. Let $\gamma : [0, 1] \to C_G(S_r)$ be a path from $y_r$ to an element $z_r \in Z(G)$. Define

$$\eta(t) = (y_1, ..., y_{r-1}, \gamma(t), y_{r+1}, ..., y_n)$$

Note that $\eta(t) \in \mathrm{Hom}(\Gamma_K, G)$ since $\gamma(t)$ commutes with $y_j$ for any $j \in S_r$. Thus $\eta$ is a path from $y$ to an element of $Z(G)^r \times G^{n-r}$, as desired. Thus by induction, $x$ has a path to some $y \in Z(G)^n$.

Let $z_1 = y$. Let $\gamma_i : z_i \rightsquigarrow z_{i+1}$ be a path that sends the $i$th component to 1 and fixes the other components. Since all other components are in $Z(G)$, the $i$th component commutes with the components of adjacent vertices, so $\gamma_i(t) \in \mathrm{Hom}(\Gamma_K, G)$. Thus the composition of the $\gamma_i$s is a path from $y$ to $(1, ..., 1)$. ☐

## 3 SO(3)-representations

Let $G = \mathrm{SO}(3)$, so that $\pi_1(G) = \{\pm 1\} \subseteq S^3$.

**Lemma 3.1.** Let $a, b \in S^3$. Then $[a, b] = -1$ iff $a, b \in S^2$ and $a \perp b$.

**Lemma 3.2.** Let $a, b \in S^2 \subseteq S^3$. Then $[a, b] = 1$ iff $a = \pm b$.

**Proposition 3.3.** Let $(K, y)$ be a marked graph, and let $(v, w)$ be a 1-marked edge whose endpoints have $-1$-marked edges. Let $K'' := K/\{v, w\}$ and $q : K \to K''$ denote the quotient map. Then

a. If $y$ has a reduction $y''$, then $y''$ is unique and

$$q^* : o_{K''}^{-1}(y'') \to o_K^{-1}(y)$$

is a homeomorphism.

b. Otherwise, $o_K^{-1}(y)$ is empty.

*Proof.* (a): Since $q$ is surjective on vertices, injectivity of $q^*$ is immediate. For surjectivity, let $x \in o_K^{-1}(y)$. Since $v$ and $w$ have $-1$-marked edges, $\widetilde{x_v}, \widetilde{x_w} \in S^2$. Then $[\widetilde{x_v}, \widetilde{x_w}] = 1 \implies \widetilde{x_v} = \pm\widetilde{x_w} \implies x_v = x_w$. So the lift $x''_{q(v)} = x_v$ is well-defined. Any continuous bijection between compact spaces is a homeomorphism, so $q*$ is a homeomorphism.

(b): Since no reduction exists, $\exists d, e \in E$ such that $q(d) = q(e)$ but $y_e \neq y_d$. WLOG let $d = (v, z)$ and $e = (w, z)$. FTSOC suppose $\exists x \in o^{-1}(y)$. Then $\widetilde{x_v} = \pm\widetilde{x_w}$ so $[\widetilde{x_v}, \widetilde{x_z}] = [\widetilde{x_w}, \widetilde{x_z}] \neq [\widetilde{x_v}, \widetilde{x_z}]$, a contradiction. ☐

**Proposition 3.4.** Let $(K, y)$ be a marked graph, and let $v \in K$ such that for all $w \in N_v$, $y_{(v,w)} = 1$. Then Let $K' = K \setminus \{v\}$, $i : K' \to K$ denote the inclusion. Then

$$i^* : o_K(y) \to o_{K'}(i^* y)$$

induces a bijection on $\pi_0$.

*Proof.* Since $i^*$ is surjective and proper, it suffices to show that the fibers are connected. Let $x \in o_K(y)$. By lemma 2.3, there is a path $\gamma : x_v \rightsquigarrow 1$ within the image of $\bigcap_{w \in N_v} C_{S^3}(\widetilde{x_w})$. So the path $\eta$ which is $\gamma$ on the $v$th component and fixes all other components is in $o_K(y)$. So $\eta : x \rightsquigarrow (z_w = x_w, z_v = 1)$, so the fiber is connected. ☐

**Proposition 3.5.** Let $(K, y)$ be a marked graph satisfying

a. all edges are marked with -1

b. there is a sequence $v_1, ..., v_n \in V$ such that $N_{v_i} \cap \{v_1, ..., v_{i-1}\}$ has cardinality $\leq 2$ for all $i$

Then $o_K^{-1}(y)$ is connected.

*Proof.* By induction on $|V|$. Pick $v_0 \in V$, and let $K' = K \setminus \{v_n\}$. By induction, $K'$ satisfies (b). Consider the surjection

$$i^* : o_K(-1) \to o_{K'}(-1)$$

The fiber is

$$(i^*)^{-1}(x) \cong \{x_n \mid [\widetilde{x_n}, \widetilde{x_i}] = -1 \text{ for all } i \in N_n\} = \pi\Big(\bigcap_{i \in N_n} C_G(\widetilde{x}_i, -1)\Big)$$

where $\pi$ is the projection $S^3 \to \mathrm{SO}(3)$. But $|N_n| \leq 2$ and $C_G(\widetilde{x}_i, -1)$ is a circle in $S^2$, so $\bigcap_{i \in N_n} C_G(\widetilde{x}_i, -1)$ is

- $S^3$ if $|N_n| = 0$
- $S^1$ if $|N_n| = 1$

- $S^1$ or two antipodal points if $|N_n| = 2$

In all cases, the image in SO(3) is connected, so the fiber $(i^*)^{-1}(x)$ is connected. Thus $i^*$ is a proper map with connected fibers and base, so the domain is connected. □

**Theorem 3.6.** Let $K$ be a disjoint union of cycles, trees, and complete graphs. Then the map

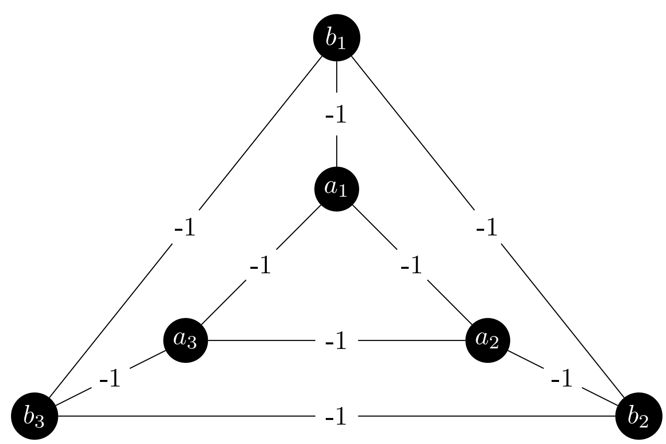
$$o : \pi_0(\mathrm{Hom}(\Gamma_K, \mathrm{SO}(3))) \to \mathbb{Z}_{/2}^E$$

is injective.

*Proof.* Consider a marking $y$. We claim that $o^{-1}(y)$ is connected. By propositions 3.3 and 3.4, we can reduce to the case where $y = (-1)_{v\in V}$ and $K'$ is a full subgraph or quotient graph of $K$. But any quotient or full subgraph of a complete graph is a complete graph, and any quotient or full subgraph of a cycle or tree is a disjoint union of cycles and trees.

Since $o^{-1}_{K_1 \amalg K_2}((y_1, y_2)) = o^{-1}_{K_1}(y_1) \times o^{-1}_{K_2}(y_2)$, it suffices to suppose $K'$ is connected. For cycles and trees, proposition 3.5 implies that $o^{-1}(y)$ is connected. For complete graphs, if $|V| \geq 4$ then $o^{-1}(y)$ is empty, otherwise by proposition 3.5 it is connected. □

**Remark 3.7.** $o$ is not injective for all graphs. For example, consider the marked graph $(M, y)$




Suppose $(a_1, a_2, a_3, b_1, b_2, b_3) \in o^{-1}(y)$. Then either

a. $a_2 = b_1$. Then $a_3 \neq a_2 = b_1$, so since $a_1$ and $b_3$ are both perpendicular to both $a_3$ and $b_1$, $b_3 = a_1$. By the same logic, $b_2 = a_3$.

b. $a_2 \neq b_1$. Then $a_1 = b_2$, $a_2 = b_3$, and $b_3 = a_1$.

So the fiber $o^{-1}(y)$ has two components: one with $(b_1, b_2, b_3) = (a_2, a_3, a_1)$ and one with $(b_1, b_2, b_3) = (a_3, a_1, a_2)$. A similar construction yields marked graphs with 3-component and $2^n$-component fibers. (todo)

**Theorem 3.8.** Let $K$ be a disjoint union of cycles and trees. Then the map

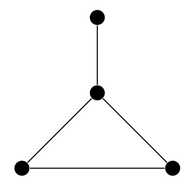
$$o : \mathrm{Hom}(\Gamma_K, \mathrm{SO}(3)) \to \mathbb{Z}_{/2}^E$$

is surjective. Conversely, if $o$ is surjective, then every non-cyclic component of $K$ contains no 3-cycles.

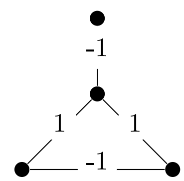
*Proof.* Since $o^{-1}_{K_1 \amalg K_2}((y_1, y_2)) = o^{-1}_{K_1}(y_1) \times o^{-1}_{K_2}(y_2)$, WLOG suppose $K$ is connected.

Suppose $K$ is a cycle or tree. We claim that $o^{-1}(y)$ is nonempty. By proposition 3.3 and 3.4, we can reduce to $y = (-1)_{v\in V}$. By proposition 3.5, $o^{-1}(y)$ is nonempty.

Conversely, suppose $K$ is not a cycle, but contains a 3-cycle. Then $K$ contains a subgraph of the form



Consider the marking $y'$ given by




Then $o^{-1}(y')$ is empty. (todo) Now pick a reduction $y$ of the marking $y'$. Since there is a map $o_K^{-1}(y') \to o^{-1}(y)$, $o^{-1}(y')$ is empty. $\square$

# 4 SO(4)-representations

Let $G = \mathrm{SO}(4)$ so that $\pi_1(G) = \{\pm 1\} \subseteq S^3 \times S^3$. Let $p$ denote the universal covering $S^3 \times S^3 \to \mathrm{SO}(4)$. For an element $\widetilde{x} \in S^3 \times S^3$, $(\widetilde{x})_1$ denotes the first component and $(\widetilde{x})_2$ denotes the second component.

**Definition 4.1.** Construct an *involution map* (todo) $\sigma : \mathrm{SO}(4) \to \mathrm{SO}(4)$ as follows: for $x \in \mathrm{SO}(4)$, pick a lift $(y, z) \in S^3 \times S^3$. Then

$$\sigma(x) := p(y, -z)$$

*Proof of well-definedness.* Let $(y, z)$ and $(y', z')$ be distinct lifts, so that $(y', z') = (-y, -z)$. Then

$$p(y, -z) = p(-y, z) = p(y', -z')$$

as desired. $\square$

**Lemma 4.2.** Let $x, z \in \mathrm{SO}(4)$. Then

a. $[\widetilde{x}, \widetilde{z}] = 1 \iff [\widetilde{\sigma(x)}, \widetilde{z}] = 1$

b. $[\widetilde{x}, \widetilde{z}] = -1 \iff [\widetilde{\sigma(x)}, \widetilde{z}] = -1$

*Proof.* Suffices to show that for $x, z \in S^3$,

$$[x, z] = [-x, z]$$

But $-1 \in Z(S^3)$. $\square$

**Proposition 4.3.** Let $(K, y)$ be a marked graph, and let $(v, w)$ be a 1-marked edge whose endpoints have -1-marked edges. Let $K'' = K/\{v, w\}$ and $q : K \to K''$ denote the quotient map. Then

a. If $y$ has a reduction $y''$, then

$$o_K^{-1}(y) \cong o_{K''}^{-1}(y) \amalg o_{K''}^{-1}(y)$$

where one component is the image of $q^*$ and the other component is the image of $\sigma_w \circ q^*$ where $\sigma_v$ is $\sigma$ on the $v$th component and the identity on other components.

b. Otherwise, $o_K^{-1}(y)$ is empty.

*Proof.* For (a), let $x \in o_K^{-1}(y)$. Pick lifts $\widetilde{x_v} = (a_v, b_v)$ and $\widetilde{x_w} = (a_w, b_w)$. Since $v$ and $w$ are adjacent to $-1$-marked edges, $a_v, b_v, a_w, b_w$ anticommute with some other element, so are in $S^2$. Thus $[a_v, a_w] = [b_v, b_w] = 1$, so $a_v = ka_w$ and $b_v = lb_w$ for $k, l \in \pm 1$. We can choose a lift such that $k = 1$, so the cases are $l = 1, -1$.

Let $x'' \in o_{K''}^{-1}(y)$ be given by $(x'')_{q(u)} = x_u$ for $u \neq v, w$, and $(x'')_{q(v)} = x_v$. If $l = 1$, then $x = q^*x''$; otherwise, $x = \sigma_v(q^*x'')$, as desired. So $\operatorname{im} q^*$ and $\operatorname{im} \sigma_v \circ q^*$ cover the $o_K^{-1}(y)$. Conversely, $\operatorname{im} q^*$ and $\operatorname{im} \sigma_w \circ q^*$ are disjoint since points in the former have $k = l$ and points in the latter have $k \neq l$.

Since $q$ is a proper injection, $q$ is a topological embedding, and since $\operatorname{im} q^*$ and $\operatorname{im} \sigma_w \circ q^*$ are disjoint closed sets, the homeomorphism holds.

For (b), since no reduction exists, $\exists d, e \in E$ such that $q(d) = q(e)$ but $y_d \neq y_e$. WLOG let $d = (v, z)$ and $e = (w, z)$. FTSOC suppose $\exists x \in o_K^{-1}(y)$. Then $\widetilde{x_v} = (\pm 1, \pm 1)\widetilde{x_w}$ so $[\widetilde{x}_v, \widetilde{x}_z] = [\widetilde{x}_w, \widetilde{x}_z] \neq [\widetilde{x}_v, \widetilde{x}_z]$, a contradiction. ☐

**Proposition 4.4.** Let $(K, y)$ be a marked graph, and let $v \in K$ be a vertex such that all neighboring edges are 1-marked. Let $K' = K \smallsetminus \{v\}$, let $i : K' \to K$ denote the inclusion. Suppose in every connected component of $o_{K'}^{-1}(i^*y)$, $\exists x'$ such that $[(\widetilde{x'_w})_1, (\widetilde{x'_u})_1] = 1$ for all $w, u \in N_v$. Then

$$i^* : o_K^{-1}(y) \to o_{K'}^{-1}(i^*y)$$

is surjective and induces a bijection on $\pi_0$.

*Proof.* Let $x' \in o_{K'}(i^*y)$. Note that any element of $(i^*)^{-1}(x)$ can be identified by the $v$th component. So

$$(i^*)^{-1}(x) \cong \{z \in \mathrm{SO}(4) \mid [\widetilde{z}, \widetilde{x_w}] = 1 \text{ for all } w \in N_v\}$$

But that's just the image of $\bigcap_{w \in N_v} C_{S^3 \times S^3}(\widetilde{x_w})z$.

If for some $j = 1, 2$ we have $[(\widetilde{x'_w})_j, (\widetilde{x'_u})_j] = 1$ for all $w, u \in N_v$, then all $\widetilde{x'_w}$ lie a common great circle passing through 1. WLOG let $j = 1$, so $\bigcap_{w \in N_v} C_{S^3 \times S^3}(\widetilde{x_w})$ is $S^3 \times D$ or $S^1 \times D$, where $D = S^3, S^1, S^0$. In any of these cases, the image in $\mathrm{SO}(4)^V$ is connected. Conversely, if for all $j = 1, 2$, there exists $w, u \in N_v$ with $[(\widetilde{x'_w})_j, (\widetilde{x'_u})_j] \neq 1$, then

$$\bigcap_{w \in N_v} C_{S^3 \times S^3}(\widetilde{x_w}) = S^0 \times S^0$$

whose image is $\{p(1, 1), p(1, -1)\}$, a discrete set with 2 points. Let $U$ denote this set of $x$s.

It suffices to show for a connected component $C$ that $(i^*)^{-1}(C)$ is connected. Consider $U_C := U \cap C$, and let $Z_C := C \smallsetminus U$. By assumption, $Z_C$ is nonempty. Let

$$A_C = \{x \mid i^*(x) \in C \text{ and } \widetilde{x_v} = (1, 1)\}$$
$$B_C = \{x \mid i^*(x) \in C \text{ and } \widetilde{x_v} = (1, -1)\}$$

and note that $A_C, B_C \cong C$. Furthermore, note that $U_C \subsetneq A_C \cup B_C$, and $A_C$ and $B_C$ both intersect $Z_C$.

Since $i^* : Z_C \to Z$ has connected fibers, it induces a bijection on $\pi_0$, so every connected component of $Z_C$ intersects $A_C$ and $B_C$. So $Z_C \cup A_C$ is connected, and thus $Z_C \cup A_C \cup B_c = Z_C \cup U_C = (i^*)^{-1}(C)$ is connected. ☐

**Lemma 4.5.** Let $K$ be a tree, $v$ be a vertex of $K$. Then $K' := K \smallsetminus \{v\}$ is a disjoint union of trees, and $N_v$ contains at most one vertex from each component of $K'$.

**Lemma 4.6.** Let $(K_i, y_i)$ be a marked tree, $(K, y) = \coprod (K_i, y_i)$. Let $N \subsetneq V$ contain at most one vertex from each component of $K$. Then in every connected component of $o_K^{-1}(y)$, $\exists x$ such that $\forall u, w \in N$, $[(\widetilde{x_w})_1, (\widetilde{x_u})_1] = 1$.

*Proof.* First, suppose $K$ has no vertex whose neighboring edges are all 1-marked. Note

$$o_K^{-1}(y) \cong \prod_i o_{K_i}^{-1}(y_i)$$

Let $x = (x_1, ..., x_r) \in o_K^{-1}(y)$, where $x_i$ denotes the component in $o_{K_i}^{-1}(y_i)$.

Suppose $\exists v_i \in N \cap V_i$. By assumption, $v_i$ is unique. Let $z_i = (\widetilde{v_i})$. Since the conjugation action of $S^3$ acts transitively on $\Re^{-1}(\Re(z_i)) \cong S^2$, $\exists g \in S^3$ such that $gz_ig^{-1} \in S^1 \subseteq \mathbb{C}$. Let $a_i = \overline{(g,1)}x_i\overline{(g,1)}^{-1}$. If $\nexists v_i \in N \cap V_i$, let $a_i = x_i$. Now let

$$a = (a_1, ..., a_r)$$

By construction, for any $v \in N$, $(\widetilde{a_v})_1 \in S^1 \subseteq \mathbb{C}$, so all $(\widetilde{a_v})_1$ commute. Furthermore, since $S^3$ is connected and conjugation is continuous, there is a path from $x_i$ to $a_i$ within $o_{K_i}^{-1}(y_i)$, so there is a path from $x$ to $a$ within $o_K^{-1}(y)$.

Now suppose $K$ has $m$ vertices whose neighboring edges are all 1-marked. Let $v$ be one such vertex. By induction on $m$, $K' := K \setminus \{v\}$ has the property that in every component $C$ of $o_{K'}^{-1}(i^*y)$, $\exists x_C \forall u, v[(\widetilde{x_w})_1, (\widetilde{x_u})_1] = 1$.

- First suppose $v \notin N$. By the previous proposition, the map $i^* : o_K^{-1}(y) \to o_{K'}^{-1}(i^*y)$ is surjective and induces an isomorphism on $\pi_0$, so for every path component $C$, lift $x_C$ along $i^*$, so that $\widetilde{x_C}$ satisfies the desired property for $(i^*)^{-1}(C)$.

- Now suppose $v \in N$. By the previous proposition, the map $i^* : o_K^{-1}(y) \to o_{K'}^{-1}(i^*y)$ is surjective and induces an isomorphism on $\pi_0$, so for every path component $C$, lift $x := x_C$ along $i^*$. Furthermore, by proposition 2.(todo) , there is a path from $(\widetilde{x_v})_1 \to \pm 1 := a_v$ within the centralizers of $\widetilde{x}_{w \in N_v}$. Thus there is a path from $x$ to $a$ where

$$a_w = \begin{cases} a_v & w = v \\ x_w & w \neq v \end{cases}$$

  and $a$ satisfies the desired property.

∎

**Lemma 4.7.** Let $(K, y)$ be a marked line with at least two $-1$-marked edges, and let $N$ be the endpoints of $K$. Then in every connected component of $o_K^{-1}(y)$, $\exists x$ such that $[(\widetilde{x_v})_1, (\widetilde{x_w})_1] = 1$ for any $v, w \in N$.

*Proof.* WLOG label $V = \{1, ..., n\}$, where $(i, i+1) \in E$, so that $N = \{1, n\}$. Let $x \in o_K^{-1}(y)$, let $C$ denote the connected component of $x$.

Consider the equivalence relation $v \sim v+1$ iff $(v, v+1)$ is 1-marked, and let $W = V/\sim$. By assumption, there are at least 3 equivalence classes in $W$. $W$ has a natural ordering by $[v] < [w] \iff v < w$. For a class $[v]$, let $\rho[v]$ denote the predecessor of $[v]$ under that ordering. Define a map

$$\phi : W \to \{i, j, k\}$$

by

$$\phi([v]) = \begin{cases} i & [v] = [1], [n] \\ j & [v] \neq [1], [n] \text{ and } \phi(\rho[v]) \neq j \\ k & [v] \neq [1], [n] \text{ and } \phi(\rho[v]) = j \end{cases}$$

Note that $\phi(\rho[v]) \perp \phi([v])$.

Now we inductively construct a sequence $a^r \in C$ such that $(\widetilde{a_v^r})_1 \in \langle 1, \phi([v]) \rangle$ for all $v \leq r$. First, let $g \in S^3$ such that $g(\widetilde{x_1})_1 g^{-1} \in \langle 1, i \rangle$. Let $\gamma : 1 \rightsquigarrow g$ in $S^3$, and define

$$\eta(t) = p(\gamma(t), 1) \cdot x \cdot p(\gamma(t), 1)^{-1}$$

Since conjugation preserves commutators equal to $\pm 1$, $\eta(t) \in o_K^{-1}(y)$, and thus $\eta(t) \in C$. Let $a^1 = \eta(1)$.

For $r > 1$, the construction is as follows:

- Suppose $(r-1, r)$ is 1-marked, and $(\widetilde{a_{r-1}^{r-1}})_1 \notin Z(S^3)$. Then since $[(\widetilde{a_{r-1}^{r-1}})_1, (\widetilde{a_r^{r-1}})_1] = 1$, we have that $(\widetilde{a_r^{r-1}})_1 \in \langle 1, \phi(r) \rangle = \langle 1, \phi(r-1) \rangle$. So let $a^r := a^{r-1}$.

- Suppose $(r, r-1)$ is 1-marked, and $(\widetilde{a_r^{r-1}})_1 \in Z(S^3)$. Note $\exists g \in S^3$ with $g(\widetilde{a_r})_1 g^{-1} \in \langle 1, \phi(r)\rangle$, and let $\gamma : 1 \rightsquigarrow g$. Now consider the path

$$\eta(t)_v = \begin{cases} a_v^{r-1} & v < r \\ p(\gamma(t), 1) \cdot a_v^{r-1} \cdot p(\gamma(t), 1)^{-1} & v \geq r \end{cases}$$

  Since conjugation preserves commutators equal to $\pm 1$, $o_K(\eta(t))$ agrees with $y$ on all edges other than $(r-1, r)$. For $(r-1, r)$, since $(\widetilde{a_{r-1}^{r-1}})_1 = (\widetilde{\eta(t)})_1 \in Z(G)$, $[(\widetilde{\eta(t)_{r-1}})_1, (\widetilde{\eta(t)_r})_1] = 1$, so $\eta(t) \in o_K^{-1}(y)$. So let $a^r := \eta(1)$.

- Suppose $(r, r-1)$ is -1-marked. Note that $(\widetilde{a_{r-1}^{r-1}})_1 = \phi(r-1) \perp (\widetilde{a_r^{r-1}})_1, \phi(r)$. Let $\mathrm{SO}(2)_z$ denote the subgroup of rotations in SO(3) that fix $z := (\widetilde{a_{r-1}^{r-1}})_1$. By transitivity of $\mathrm{SO}(2)_z \curvearrowright S^3 \cap z^\perp$, $\exists g \in \mathrm{SO}(2)_z$ such that $g((\widetilde{a_r^{r-1}})_1) = \phi(r)$. Since $\mathrm{SO}(2)_z$ is connected, there is a path $\gamma : 1 \rightsquigarrow g$ within $\mathrm{SO}(2)_z \subseteq \mathrm{SO}(3)$, lifting to a path $\widetilde{\gamma} : 1 \rightsquigarrow \widetilde{g}$ within $S^3$ for some lift $\widetilde{g}$.

  Let $\eta : [0, 1] \to C$ be a path by

$$\eta(t)_v = \begin{cases} a_v^{r-1} & v < r \\ p(\widetilde{\gamma}(t), 1) \cdot a_v^{r-1} \cdot p(\widetilde{\gamma}(t), 1)^{-1} & v \geq r \end{cases}$$

  Note that $o_K(\eta(t))$ agrees with $y$ on all edges other than $(r-1, r)$. For $(r-1, r)$, since $\widetilde{\gamma}(t)$ fixes $z^\perp$, $(\widetilde{\eta(t)_r})_1 \perp z$, so $[z = (\widetilde{\eta(t)_{r-1}})_1, (\widetilde{\eta(t)_r})_1] = -1$, so $\eta(t) \in o_K^{-1}(y)$. Let $a^r := \eta(1)$.

Now $(\widetilde{a_n^n})_1, (\widetilde{a_1^n})_1 \in \langle 1, i\rangle$ and thus commute, so $a^n$ is the desired $x$. □

**Proposition 4.8.** Let $(K, y)$ be a marked tree with $y = -1$ on all edges. Then $o_K^{-1}(y)$ is connected.

*Proof.* By induction on $|V|$. Let $K' = K \smallsetminus \{v\}$ for some leaf $v$. Let $w$ be the unique vertex connected to $v$. Consider the map

$$i^* : o_K^{-1}(y) \to o_{K'}^{-1}(y')$$

By induction, the base is connected, so since $i^*$ is proper it suffices to show that the fibers of $i^*$ are nonempty and connected. But the fiber $(i^*)^{-1}(x')$ can be identified with

$$\{z \in \mathrm{SO}(4) \mid [\widetilde{z}, \widetilde{x'_w}] = -1\} \cong p(C_{S^3 \times S^3}(\widetilde{x'_w}, -1))$$

But $C_{S^3 \times S^3}(\widetilde{x'_w}, -1) \cong C_{S^3}((\widetilde{x'_w})_1, -1) \times C_{S^3}((\widetilde{x'_w})_2, -1)$ which is a product of $S^1$ and $S^3$ss and thus connected. □

**Theorem 4.9.** Let $(K, y)$ be a disjoint union of marked trees. Let $m$ denote the number of 1-marked edges whose endpoints have $-1$-marked edges. Then $o_K^{-1}(y)$ has $2^m$ connected components.

*Proof.* By proposition 4.3, it suffices to show that if $m = 0$ then $o_K^{-1}(y)$ is connected. Let $r$ denote the number of vertices whose edges are all 1-marked. We proceed by induction on $r$. The base case $r = 0$ is proposition 4.8. Let $v$ be one such vertex, let $K' = K \smallsetminus \{v\}$, $y' = i^* y$ where $i : K' \to K$. By inductive hypothesis, $o_{K'}^{-1}(y')$ is connected. Note that $K'$ is a disjoint union of trees, and $N_v$ contains at most one vertex from each component. By lemma 4.6 and proposition 4.4, $o_K^{-1}(y) \to o_{K'}^{-1}(y')$ induces a bijection on connected components, so $o_K^{-1}(y)$ is connected. □

# 5 $\mathrm{PSL}(2, \mathbb{C})$-representations

Let $\omega_n$ denote a primitive $n$th root of unity.

**Lemma 5.1.** $C_{\mathrm{SL}(n,\mathbb{C})}(B, \omega_n)$ is connected or empty.

*Proof.* Let $A \in C_{\mathrm{SL}(n,\mathbb{C})}(B,\omega)$. By lemma,

$$A = P' \operatorname{diag}\{\lambda, ..., \omega^{n-1}\lambda\} P'^{-1}$$

with $\lambda^n = \prod_i \omega^i = (-1)^{n+1}$. Thus $\lambda$ is an $n$th root of unity or odd $2n$th root of unity, so $A$ has the form $P \operatorname{diag}\{1, ..., \omega^{n-1}\} P^{-1}$ or $P \operatorname{diag}\{\omega_{2n}, ..., \omega_{2n} \cdot \omega^{n-1}\} P^{-1}$ with $Bp_i \in \langle p_{i+1} \rangle$. Let $D = \operatorname{diag}\{\omega_{2n}, ..., \omega_{2n} \cdot \omega^{n-1}\}$ or $\operatorname{diag}\{1, ..., \omega^{n-1}\}$. Conversely, any matrix of the above form is in $C_{\mathrm{SL}(n,\mathbb{C})}(B,\omega)$. So

$$C_{\mathrm{SL}(n,\mathbb{C})}(B,\omega) = \{PDP^{-1} \mid Bp_i \in \langle p_{i+1} \rangle \text{ for } i = 1, ..., n\}$$

Consider the map

$$\phi : \ker(B^n - I) \setminus \bigcup_{m=1}^{n-1} \ker(B^m - I) \to C_{\mathrm{SL}(n,\mathbb{C})}(B,\omega) \qquad v \mapsto P_v D P_v^{-1}$$

where $P_v = \begin{bmatrix} v & Bv & ... & B^{n-1}v \end{bmatrix}$. Since $v, Bv, ..., B^{n-1}v$ are distinct, $P_v$ is invertible, so the map is well-defined. For surjectivity, any $PDP^{-1} \in C_{\mathrm{SL}(n,\mathbb{C})}(B,\omega)$ has $p_0, Bp_0, ..., B^{n-1}p_0$ distinct, so $p_0 \in (B^n - I) \setminus \bigcup_{m=1}^{n-1} \ker(B^m - I)$ and $\phi(p_0) = PDP^{-1}$. □

**Lemma 5.2.** Let $n$ be odd (resp. even). Then $C_{\mathrm{SL}(n,\mathbb{C})}(B,\omega_n)$ is nonempty iff the eigenvalues of $B$ are $1, \omega, ..., \omega^{n-1}$ (resp. $\omega_{2n}, ..., \omega_{2n}\omega_n^{n-1}$).

*Proof.* Suppose $\exists A \in C_{\mathrm{SL}(n,\mathbb{C})}(B,\omega_n)$, then $[A,B] = \omega \implies [B,A] = \omega^{-1}$. So by lemma we have $A(E_{\lambda,B}) = E_{\omega^{-1}\lambda,B}$ so $B$ has eigenvalues $\lambda, \lambda\omega, ..., \lambda\omega^{n-1}$. But $\prod_{i=0}^{n-1} \lambda\omega^i = \lambda^n = (-1)^{n+1}$, so $\lambda$ is an $n$th root of unity (resp. odd $2n$th root of unity).

For the backwards direction, let $v_i \in E_{\omega^i,B}$ have norm 1. Then construct

$$A = PFP^{-1}$$

where $P = \begin{bmatrix} v_0 & ... & v_{n-1} \end{bmatrix}$ and $F$ has 1s in the $(i, i+1)$th entries and $(-1)^{n+1}$ in the $(n,1)$th entry. Then $\det A = 1$ and $A(v_i) = \pm v_{i+1}$, so by lemma $[A,B] = \omega$. □

**Lemma 5.3.** The image of the space of matrices in $C_{\mathrm{SL}(n,\mathbb{C})}(B)$ with eigenvalues $1, ..., \omega^{n-1}$ for $n$ odd, or $\omega_{2n}, ..., \omega_{2n}\omega_n^{n-1}$ for $n$ even, has at most $(n-1)!$ components. If $B$ is diagonalizable, then it has exactly $(n-1)!$ components.

*Proof.* The space of $n$ distinct 1-dimensional subspaces fixed by $B$ is the space of subspaces contained in eigenspaces of $B$. □

**Theorem 5.4.** Let $\Gamma_K$ be the RAAG associated to a tree. Then the map

$$o_2 : \operatorname{Hom}(\Gamma_K, \mathrm{PSL}(2,\mathbb{C})) \to \mathbb{Z}_{/2}^E$$

is injective on connected components.

*Proof.* (todo) □