Approaches to the L'vov Kaplansky Conjecture: Technical Report

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REU 2025

Abstract

The L'vov-Kaplansky Conjecture is a longstanding open question in the field of Algebra, which asks if the image of a multilinear polynomial acting on the space of $n \times n$ matrices is a subspace. We consider a weakening of the conjecture using an analytic lens: the Density Dimension-Free L'vov-Kaplansky Conjecture. This asks whether any multilinear polynomial eventually attains density in a subspace for large enough matrices with complex entries. Using an auxiliary map, we prove that density of the image on 2×2 matrices is guaranteed by a linear independence condition. Moreover, we give conditions on factorizations of multilinear polynomials that, if satisfied, guarantee density of the image within the traceless matrices or full matrix algebra. We provide concrete open problems that would shed further insight into the conjecture.

1 Introduction and Backgound

Noncommutative functions show interactions between variables that do not commute with each other. The theory surrounding these objects is based in the idea that order matters for noncommuting variables. For example, $n \times n$ matrices with elements in the complex numbers (denoted $M_n(\mathbb{C})$) are not commutative with each other under standard matrix multiplication. We can illustrate this by an example:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 8 & 5 \\ 20 & 13 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 13 & 20 \\ 5 & 8 \end{pmatrix}.$$

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The desire to understand the structure of matrices stems from many fields in the sciences, notably engineering and physics. This summer, we aim to generalize results related to how matrices act under special objects called multilinear polynomials in relation to the L'vov-Kaplansky Conjecture.

Conjecture 1.1 (L'vov-Kaplansky)**.** *If p be a multilinear polynomial, then the image of p on $M_n(\mathbb{C})$ is a subspace.*

1.1 Multilinear Polynomials

Consider a polynomial in several noncommuting variables $(X_1, ..., X_d)$ with no relations between them. We denote this by $p \in \mathbb{C}\langle X_1, ..., X_d\rangle$, where $\mathbb{C}\langle X_1, ..., X_d\rangle$ is the free algebra generated by $X_1, ..., X_d$. For example, we can define p_1 by

$$p_1(X_1, X_2) = X_1^2 + 3X_2 + 2.$$

For another example, consider p_2 by

$$p_2(X_1, X_2, X_3) = X_1X_2X_3.$$

We are particularly interested in the *image* of these polynomials evaluated on $n \times n$ matrices, defined as follows:

Definition 1.2. The *image on $M_n(\mathbb{C})$* of a polynomial $p \in \mathbb{C}\langle X_1, ..., X_d\rangle$ is the set

$$\mathrm{im}_n(p) = \{p(Z) \in M_n(\mathbb{C}) : Z \in M_n(\mathbb{C})^d\}.$$

For our study, we are not interested in all polynomials, but particular types. The L'vov-Kaplansky Conjecture focuses on objects known as multilinear polynomials. A more general type of polynomial known as homogeneous polynomials will also be briefly explored.

Definition 1.3. A polynomial $p(X_1, \cdots, X_n)$ is *homogeneous of degree d* if for all λ,

$$p(\lambda X_1, ..., \lambda X_n) = \lambda^d p(X_1, \cdots, X_n).$$

Importantly, homogeneous polynomials scale nicely when each variable is scaled. Take, for instance,

$$p(X_1, X_2) = X_1^2 + X_1X_2,$$

which is homogeneous of degree 2. Note that $p(3X_1, 3X_2) = 9X_1^2 + 9X_1X_2 = 9p(X_1, X_2) = 3^2p(X_1, X_2)$.

We can also require that our polynomial be linear in each variable, which can add more structure.

Definition 1.4. A polynomial $p(X_1, \cdots, X_d)$ is *multilinear* if each term in p contains exactly each variable exactly once. Every such polynomial p has the form

$$p(X_1, ..., X_d) = \sum_{\sigma \in S_d} \alpha_\sigma X_{\sigma(1)} \cdots X_{\sigma(d)}.$$

We list some examples of multilinear polynomials below:

- $p_1(X, Y) = XY - YX$
- $p_2(X, Y) = 2XY + 6YX$
- $p_3(X, Y, Z, W) = 3XYZW$
- $p_4(X, Y, Z) = XYZ + YZX + 2XZY + iYXZ + ZYX + ZXY$

Some nonexamples of multilinear polynomials include:

- $p_5(X, Y, Z) = XY + YZ$
- $p_6(X, Y) = X^2 + Y^2$
- $p_7(X, Y, Z) = XYZ + ZYZ$

Multilinear polynomials have some nice properties. For one, every multilinear polynomial is homogeneous. Additionally, every multilinear polynomial is linear in each of its variables separately:

$$p(X_1 + Y_1, X_2, \cdots X_n) = p(X_1, X_2, \cdots, X_n) + p(Y_1, X_2, \cdots, X_n).$$

However, multilinear polynomials are not simultaneously linear in each of their variables.

1.2 Special Multilinear Polynomials

There are a few special types of multilinear polynomials that are important to the study of the L'vov-Kaplansky Conjecture.

Definition 1.5. A multilinear polynomial p is a *polynomial identity* on $M_n(\mathbb{C})$ if $\text{im}_n(p) = \{0\}$.

If p is a polynomial identity on $M_n(\mathbb{C})$, then p is also a polynomial identity on $M_m(\mathbb{C})$ for all $m \leq n$.

Definition 1.6. A multilinear polynomial p is the *standard polynomial* on $M_n(\mathbb{C})$ if p is of the form

$$p(X_1, ..., X_{2n}) = \sum_{\sigma \in S_{2n}} \text{sgn}(\sigma) X_{\sigma(1)} \cdots X_{\sigma(2n)}$$

.

By the Amitsur-Levitzki Theorem [1], the standard polynomial is the polynomial identity on $M_n(\mathbb{C})$ of lowest degree.

Definition 1.7. A multilinear polynomial p is a *central polynomial* for $M_n(\mathbb{C})$ if $\text{im}_n(p) \subseteq \mathbb{C} I_n$.

2 Linear Algebra Review

A well-known result of Linear Algebra states that the image of a linear map is a subspace. The L'vov-Kaplansky Conjecture asks if the same holds for multilinear polynomials. Using the framework of Sheldon Axler's textbook *Linear Algebra Done Right* [2], we reviewed topics such as vector spaces, linear independence, and similarities. Some of the relevant topics that we covered are presented below. We specifically looked at vector spaces over $\mathbb{C}$.

2.1 Vector Spaces and Subspaces

Definition 2.1. A set V is a *vector space* over $\mathbb{C}$ if it has two operations–addition and scalar multiplication–which meet the following axioms:

1. Addition is commutative: for all $x, y \in V$, we have $x + y = y + x$.
2. Addition and scalar multiplication are associative: for all $\vec{x}, \vec{y}, \vec{z} \in V$ and $a, b \in \mathbb{C}$, we have $(\vec{x} + \vec{y}) + \vec{z} = \vec{x} + (\vec{y} + \vec{z})$ and $a(b\vec{x}) = (ab)\vec{x}$.
3. There exists an additive identity 0 in V such that $0 + \vec{v} = \vec{v}$ for all $\vec{v} \in V$.
4. Every element $\vec{x} \in V$ has an additive inverse: there exists some $-\vec{v}$ such that $\vec{v} + -\vec{v} = 0$.
5. There exists a scalar $1 \in \mathbb{C}$ such that $1\vec{x} = \vec{x}$ for all $\vec{x} \in V$.
6. Addition and scalar multiplication distribute: for all $\vec{x}, \vec{y} \in V$ and $a, b \in \mathbb{C}$ we have $(a + b)\vec{x} = a\vec{x} + b\vec{x}$ and $a(\vec{x} + \vec{y}) = a\vec{x} + b\vec{y}$.

Examples of vector spaces over $\mathbb{C}$ include $\mathbb{C}^n$, $\mathbb{R}^n$, $\mathcal{P}_n(x) = \{a_0 + a_1x + \cdots + a_nx^n : a_i \in \mathbb{C}\}$. Of particular interest to this project is the $n \times n$ matrices with complex coefficients, denoted $M_n(\mathbb{C})$.

Definition 2.2. Given V, a vector space over $\mathbb{C}$ and $W \subseteq V$, then W is a *subspace* of V if W is a vector space over $\mathbb{C}$.

While verifying all of the axioms that determine if a set is a vector space are labor-intensive, given a subset of a vector space, there is a shorter list of necessities.

Theorem 2.3. *If V is a vector space over $\mathbb{C}$ and $W \subseteq V$, then W is a subspace of V if and only if*

1. *We have $0 \in W$.*
2. *If $\vec{x} \in W$, then $c\vec{x} \in W$ for all $c \in \mathbb{C}$.*
3. *For all $\vec{x}, \vec{y} \in W$, we have $\vec{x} + \vec{y} \in W$.*

When we consider the L'vov-Kaplansky conjecture, it will be the third requirement, additivity, that will pose problems for the images of multilinear polynomials. It is straightforward to show that the zero matrix is in the image of any multilinear polynomial. Also, due to the scaling of homogeneous polynomials, and thus multilinear polynomials, verifying the second condition is likewise straightforward. However, showing that additivity is satisfied is much more difficult.

2.2 Linear Independence and Span

In this section, we will always assume that V is a finite-dimensional vector space over $\mathbb{C}$.

Definition 2.4. We say that a set of vectors $\{\vec{v_1}, \vec{v_2}, ..., \vec{v_n}\} \subseteq V$ are *linearly independent* if

$$c_1\vec{v_1} + c_2\vec{v_2} + ... + c_n\vec{v_n} = 0$$

then $c_1, ..., c_n = 0$.

Linear independence will be particular useful when we consider the images of multilinear polynomials on 2×2 matrices.

Definition 2.5. The *span* of a set of vectors $\{\vec{v_1}, \vec{v_2}, ..., \vec{v_n}\} \subseteq V$ is given by

$$\text{span}\{\vec{v_1}, \vec{v_2}, ..., \vec{v_n}\} = \{c_1\vec{v_1} + c_2\vec{v_2} + ... + c_n\vec{v_n} : c_1, ..., c_n \in \mathbb{C}\}.$$

For any set of vectors $\{\vec{v_1}, \vec{v_2}, ..., \vec{v_n}\} \subseteq V$, we have that $\text{span}\{\vec{v_1}, \vec{v_2}, ..., \vec{v_n}\}$ is always a subspace of V.

Definition 2.6. We say that a set of vectors $\{\vec{v_1}, \vec{v_2}, ..., \vec{v_n}\} \subseteq V$ form a *basis* for V if $\text{span}\{\vec{v_1}, \vec{v_2}, ..., \vec{v_n}\} = V$ and $\{\vec{v_1}, \vec{v_2}, ..., \vec{v_n}\}$ are linearly independent.

Any basis for V has the same number of elements. This is the dimension of V, denoted $\dim(V)$.

2.3 Similarities

One topic of linear algebra that is particularly relevant to the L'vov-Kaplansky Conjecture is conjugation by similarities.

Definition 2.7. We say that two matrices, $A, B \in M_n(\mathbb{C})$ are *similar* if there exists an invertible matrix $S \in M_n(\mathbb{C})$ such that $A = SBS^{-1}$.

By the Schur Decomposition, we know that any matrix in $M_n(\mathbb{C})$ is similar, and actually unitarily equivalent, to an upper triangular matrix, with entries only on and above the main diagonal.

Definition 2.8. We say that a set $\Omega \subseteq M_n(\mathbb{C})$ is *invariant under conjugation by similarities* if we have $\Omega = S\Omega S^{-1}$ for all invertible matrices $S \in M_n(\mathbb{C})$

This is particularly relevant to the L'vov-Kaplansky conjecture because for any polynomial $p \in \mathbb{C}\langle x_1, ..., x_d \rangle$, and in particular, for multilinear polynomials, we have the following equality for all invertible $S \in M_n(\mathbb{C})$:

$$S^{-1}p(X_1, ..., X_d)S = p(S^{-1}X_1S, ..., S^{-1}X_dS)$$

where $(X_1, ..., X_d) \in M_n(\mathbb{C})^d$. This implies that $S^{-1}p(X_1, ..., X_d)S \in \text{im}_n(p)$ for all invertible $S \in M_n(\mathbb{C})$. Therefore, we have that $\text{im}_n(p)$ is invariant under similarities. It has been shown [4] that the only subspaces $V \subseteq M_n(\mathbb{C})$ that are invariant under conjugation by similarities are

1. The zero matrix: $\{0\}$
2. Scalar multiples of the identity matrix: $\mathbb{C}I_n$
3. Trace zero matrices: $M_n^0(\mathbb{C})$
4. The full space: $M_n(\mathbb{C})$.

This gives a slightly narrower focus to the search for subspaces as the image of multilinear polynomials. If we were to find a multilinear polynomial that did not have one of these four subspaces as its image, this would be a contradiction of the L'vov-Kaplansky Conjecture.

2.4 Eigenvalues, Determinant, and Trace

In attempting to prove the L'vov-Kaplansky Conjecture, we will utilize properties of matrices.

Definition 2.9. We say that $\lambda \in \mathbb{C}$ is an *eigenvalue* of a matrix $A \in M_n(\mathbb{C})$ if there exists a vector $\vec{v}$ such that

$$A\vec{v} = \lambda\vec{v}.$$

We call $\vec{v}$ an *eigenvector* of A.

Definition 2.10. We say that a matrix $A \in M_n(\mathbb{C})$ is *diagonalizable* if there exists an invertible matrix $S \in M_n(\mathbb{C})$ such that SAS^{-1} is a diagonal matrix.

A matrix $A \in M_n(\mathbb{C})$ is diagonalizable if A has n distinct eigenvalues. However, a matrix may be diagonalizable even if it does not have n distinct eigenvalues, an example is the $n \times n$ identity matrix, I_n. In addition to providing insight into diagonalizability, the eigenvalues of a matrix are the building blocks of the characteristic polynomial of a matrix, which gives important information about the matrix itself.

Definition 2.11. The *characteristic polynomial* of a matrix $A \in M_n(\mathbb{C})$ with distinct eigenvalues $\lambda_1, ..., \lambda_n$ and multiplicities $d_1, ..., d_n$ is the commutative polynomial

$$q(z) = (z - \lambda_1)^{d_1} ... (z - \lambda_n)^{d_n},$$

By the Caley-Hamilton Theorem, if q is the characteristic polynomial of a matrix A, then $q(A) = 0$. Other information about matrices is also related by the eigenvalues.

Definition 2.12. The *determinant* of a matrix $A \in M_n(\mathbb{C})$ is the product of the eigenvalues of A. We denote the determinant of A by $\det(A)$.

The determinant of a matrix is fairly straightforward to compute in the 2×2 case. For the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

we have the determinant is $ad - bc$. However, for larger matrix sizes, finding the determinant, while possible, is more computationally difficult. A well-known result relates the determinant of a matrix A to invertiblity.

Theorem 2.13. *A matrix $A \in M_n(\mathbb{C})$ is invertible if and only if $\det(A) \neq 0$.*

Definition 2.14. The *trace* of a matrix A is the sum of the values on the main diagonal of A. The trace of a matrix is also the sum of its eigenvalues. We denote the trace of A by $\operatorname{tr}(A)$

For example, we have that the trace of the following matrix is 9:

$$\begin{pmatrix} 4 & 6 \\ -1 & 5 \end{pmatrix}$$

We have for any $A, B \in M_n(\mathbb{C})$, $\operatorname{tr}(AB) = \operatorname{tr}(BA)$. This is true for any cyclic permutation of matrices, for example $\operatorname{tr}(XYZ) = \operatorname{tr}(ZYX)$ but $\operatorname{tr}(XYZ) \neq \operatorname{tr}(XZY)$.

Remark 2.15. For a matrix $A \in M_n(\mathbb{C})$, the determinant and trace are similarity invariant. That is, for any invertible $S \in M_n(\mathbb{C})$, we have $\det(A) = \det(SAS^{-1})$ and $\operatorname{tr}(A) = \operatorname{tr}(SAS^{-1})$.

In addition to being closely related to eigenvalues, determinant and trace are also closely related to the characteristic polynomial. By the LeVerrier Characteristic Polynomial Recursion [7], the coefficients of the characteristic polynomial of a matrix A are related to the determinant, the trace, and trace of powers of A. Thus, the determinant and trace will be useful tools in our study of the L'vov-Kaplansky Conjecture.

2.5 The L'vov-Kaplansky Conjecture

Our main area of study revolves around the previously stated L'vov-Kaplansky Conjecture, which we will state again due to its importance. For our purposes, we will consider the conjecture over the complex numbers.

Conjecture 1.1 (L'vov-Kaplansky). *If p is a multilinear polynomial, then the image of p on $M_n(\mathbb{C})$ is a subspace.*

While amazingly simple to state, the conjecture has remained unproved for decades, although partial results exist [6, 5, 13]. A relaxed conjecture explores if the L'vov-Kaplansky Conjecture is true for large enough matrix sizes.

Conjecture 2.16 (Dimension-Free L'vov-Kaplansky)**.** *If p is a multilinear polynomial, then there exists an $N \in \mathbb{N}$ such that $n \geq N$ implies that $\operatorname{im}_n(p)$ is a subspace.*

We propose another relaxation that allows us to tackle the L'vov-Kaplansky Conjecture from an analytical approach.

Conjecture 2.17 (Density Dimension-Free L'vov-Kaplansky Conjecture)**.** *If p is a multilinear polynomial, then there exists $N \in \mathbb{N}$ so that for $n \geq N$, $\operatorname{im}_n(p)$ is dense in a subspace of $M_n(\mathbb{C})$.*

We present two different definitions of density below. We note that for our purposes, all statements will remain true if we replace Zariski density with Euclidean density [12].

Definition 2.18. A set $S \subseteq \mathbb{C}^n$ is *Euclidean dense* if every point $z \in \mathbb{C}^n$ is arbitrarily close to a point in S (using Euclidean distance).

Definition 2.19. A set $S \subseteq \mathbb{C}^n$ is *Zariski dense* if S is everything except possibly the zero set of a non-zero polynomial.

3 Bilinear Case

Definition 3.1. A *bilinear* polynomial is a multilinear polynomial of degree 2. Bilinear polynomials can be written as $p(X,Y) = aXY + bYX$ for some $a, b \in \mathbb{C}$.

We will show that all bilinear polynomials satisfy the full L'vov-Kaplansky Conjecture. To do this, we will first prove a classical result about a famous bilinear polynomial.

Definition 3.2. The *commutator* is the bilinear polynomial

$$p(X,Y) = XY - YX.$$

The commutator is denoted $[X,Y]$.

Theorem 3.3. *For the commutator, $p(X,Y) = [X,Y]$, we have $\operatorname{im}_n(p) = M_n^0(\mathbb{C})$.*

While this result was proved in [11], we provide a proof of this theorem in English to aid with understanding. We will utilize two lemmas to prove this theorem.

Lemma 3.4. *For the commutator, $p(X,Y) = [X,Y]$, we have all zero-diagonal matrices in $\operatorname{im}_n(p)$.*

Proof. Let $Z \in M_n(\mathbb{C})$ be an arbitrary matrix with zeroes along the main diagonal. Thus, we have

$$Z = \begin{pmatrix} 0 & z_{12} & \dots & z_{1n} \\ z_{21} & 0 & \dots & z_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ z_{n1} & z_{n2} & \dots & 0 \end{pmatrix}.$$

We will show that there exist matrices $A, B \in M_n(\mathbb{C})$ such that $Z = [A, B]$, and hence $Z \in \text{im}_n(p)$.
Let $A \in M_n(\mathbb{C})$ be a diagonal matrix with distinct entries. Hence,

$$A = \begin{pmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{pmatrix}$$

where $a_{ii} \neq a_{jj}$ for $i \neq j$. Now, we define a matrix $B \in M_n(\mathbb{C})$ entry-wise. Let

$$b_{ij} = \begin{cases} 0 & \text{if } i = j \\ \frac{z_{ij}}{a_{ii} - a_{jj}} & \text{if } i \neq j \end{cases}$$

Thus, we have

$$\begin{aligned} p(A, B) &= \begin{pmatrix} a_{11}b_{11} & a_{11}b_{12} & \dots & a_{11}b_{1n} \\ a_{22}b_{21} & a_{22}b_{22} & \dots & a_{22}b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{nn}b_{n1} & a_{nn}b_{n2} & \dots & a_{nn}b_{nn} \end{pmatrix} - \begin{pmatrix} a_{11}b_{11} & a_{22}b_{12} & \dots & a_{nn}b_{1n} \\ a_{11}b_{21} & a_{22}b_{22} & \dots & a_{nn}b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{11}b_{n1} & a_{22}b_{n2} & \dots & a_{nn}b_{nn} \end{pmatrix} \\ &= \begin{pmatrix} 0 & (a_{11} - a_{22})b_{12} & \dots & (a_{11} - a_{nn})b_{1n} \\ (a_{22} - a_{11})b_{21} & 0 & \dots & (a_{22} - a_{nn})b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ (a_{nn} - a_{11})b_{n1} & (a_{nn} - a_{22})b_{n2} & \dots & 0 \end{pmatrix}. \end{aligned}$$

Therefore, in general, we have $(p(A, B))_{ij} = (a_{ii} - a_{jj})b_{ij}$. By our choice of b_{ij}, this implies that $(p(A, B))_{ij} = z_{ij}$. Thus, we have that $p(A, B) = Z$, and consequently $Z \in \text{im}_n(p)$. Therefore, we have all zero-diagonal matrices in $\text{im}_n(p)$. □

Now, because we know that multilinear polynomials are similarity invariant, we will show that attaining all zero-diagonal matrices is equivalent to attaining all traceless matrices.

Lemma 3.5. *If $Z \in M_n^0(\mathbb{C})$, then there exists some invertible $S \in M_n(\mathbb{C})$ such that SZS^{-1} is a zero-diagonal matrix.*

Proof. Let $Z \in M_n^0(\mathbb{C})$ be arbitrary. This proof proceeds by induction on n. When $n = 1$, then $Z = \begin{pmatrix} 0 \end{pmatrix}$, which is clearly a zero-diagonal matrix. Thus, assume the claim holds for $n-1$, this is the induction hypothesis.

By the Schur Decomposition, there exists an invertible matrix S_1 such that $S_1 Z S_1^{-1}$ is an upper-triangular matrix. Let $T = S_1 Z S_1^{-1}$, then

$$T = \begin{pmatrix} t_{11} & t_{12} & \dots & t_{1n} \\ 0 & t_{22} & \dots & t_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & t_{nn} \end{pmatrix}$$

Because $\operatorname{tr}(T) = \operatorname{tr}(Z)$, we have $\operatorname{tr}(T) = 0$. Thus, $\sum_i^n t_{ii} = 0$. If $t_{ii} = 0$ for all i, we are done. Thus, assume $t_{ii} \neq 0$ for some i, and therefore we must have that $t_{ii} \neq t_{jj}$ for some i, j. Without loss of generality, assume that $t_{11} \neq t_{22}$. So, let R be the 2×2 block matrix in the upper-left hand corner of T, we have

$$R = \begin{pmatrix} t_{11} & t_{12} \\ 0 & t_{22} \end{pmatrix}.$$

Because $t_{11} \neq t_{22}$, R has distinct eigenvalues, and thus is diagonalizable. So, there exists some invertible $T_1 \in M_2(\mathbb{C})$, such that

$$Q = T_1 R T_1^{-1} = \begin{pmatrix} q_{11} & 0 \\ 0 & q_{22} \end{pmatrix}.$$

Now, consider the matrix

$$T_2 = \begin{pmatrix} 1 & t \\ -t & 1 \end{pmatrix},$$

which is invertible if $t \neq \pm i$. Then, we have

$$\begin{aligned} T_2 Q T_2^{-1} &= \frac{1}{1+t^2} \begin{pmatrix} 1 & t \\ -t & 1 \end{pmatrix} \begin{pmatrix} q_{11} & 0 \\ 0 & q_{22} \end{pmatrix} \begin{pmatrix} 1 & -t \\ t & 1 \end{pmatrix} \\ &= \frac{1}{1+t^2} \begin{pmatrix} q_{11} + q_{22}t^2 & -q_{11}t + q_{22}t \\ -q_{11}t + q_{22}t & q_{11}t^2 + q_{22} \end{pmatrix}. \end{aligned}$$

Because $\mathbb{C}$ is an algebraically closed field, there exists a solution to the equation $q_{11} + q_{22}t^2 = 0$. Let w be a solution to this equation. Because $q_{11} \neq q_{22}$, we have $w \neq \pm i$, and thus T_2 is invertible. Therefore, we have

$$T_2 Q T_2^{-1} = \begin{pmatrix} 0 & * \\ * & * \end{pmatrix}.$$

Thus, we are able to force the entry in the upper left corner to be a zero. Finally, we let

$$S_2 = \begin{pmatrix} T_1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

and

$$S_3 = \begin{pmatrix} T_2 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}.$$

We then have that, by properties of multiplication of block matrices, that

$$P = S_3 S_2 S_1 Z S_1^{-1} S_2^{-1} S_3^{-1} = \begin{pmatrix} 0 & * & \dots & * \\ * & * & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \dots & * \end{pmatrix}.$$

Because trace is invariant under similarities, we have that $\mathrm{tr}(P) = \mathrm{tr}(Z) = 0$. Thus, the $n-1 \times n-1$ block matrix in the lower-right corner of P also has trace 0, to which we can apply the induction hypothesis. Therefore, every traceless matrix Z is similar to a zero-diagonal matrix. □

These lemmas allow us to prove Theorem 3.3.

Proof. Let $Z \in M_n^0(\mathbb{C})$ be arbitrary. By Lemma 3.5, there exists some invertible $S \in M_n(\mathbb{C})$ such that SZS^{-1} is a zero-diagonal matrix. Then, by Lemma 3.4, we have $SZS^{-1} \in \mathrm{im}_n(p)$. Therefore, we have $\mathrm{im}_n(p) = M_n^0(\mathbb{C})$. □

This case allows us to prove that all bilinear polynomials satisfy the full L'vov-Kaplansky Conjecture.

Theorem 3.6. *If p is a bilinear polynomial, then $\mathrm{im}_n(p)$ is a subspace of $M_n(\mathbb{C})$.*

Proof. Suppose p is a bilinear polynomial in the form

$$p(X, Y) = aXY + bYZ$$

for some $a, b \in \mathbb{C}$.
First, in the case that $a = b = 0$, $\mathrm{im}_n(p) = \{0\}$.
Next, in the case that $a = 1$ and $b = 0$, we have $p(X, Y) = aXY$. So, for all $A \in M_n(\mathbb{C}), f(\frac{1}{a}A, I_n) = A$. From this we get that the image is $M_n(\mathbb{C})$. Similarly, this works when $b = 1$ and $a = 0$.
Now, we assume that $a, b \neq 0$. Because p is a homogeneous polynomial, we have

$$\frac{1}{a}p(X, Y) = p\left(\frac{1}{a}X, Y\right)$$

and also

$$\frac{1}{a}p(X, Y) = XY + \frac{b}{a}XY.$$

Therefore, $\mathrm{im}_n(\frac{1}{a}p) = \mathrm{im}_n(p)$, and so we can assume without loss of generality that $a = 1$. If $b \neq -1$, then $p(I_n, Y) = (1 + b)Y$. Therefore, because $1 + b \neq 0$,

for any $A \in M_n(\mathbb{C})$, we have $p(I_n, \frac{1}{1+b}A) = A$. Thus, $\text{im}_n(p) = M_n(\mathbb{C})$. Finally, if $b = -1$, then p is the commutator. By Theorem 3.3, we have that $\text{im}_n(p) = M_n^0(\mathbb{C})$.
Therefore, we have shown that all bilinear polynomials have an image of $\{0\}, M_n^0(\mathbb{C})$, or $M_n(\mathbb{C})$, all of which are subspaces of $M_n(\mathbb{C})$. □

The relative ease of the bilinear case leads to some confidence in approaching multilinear polynomials in terms of their degree to attempt to answer the conjecture. However, as we will see in the trilinear case, the number of the coefficients of a multilinear polynomial is factorial. The strategy of using the identity matrix to obtain useful information about the image of a multilinear polynomial, while still pertinent, does not lead to answers in all cases.

4 Trilinear Examples

A *trilinear* polynomial is a multilinear polynomial of degree three. We can express a trilinear polynomial p as

$$p(X, Y, Z) = aXYZ + bXZY + cYXZ + dYZX + eZXY + fZYX,$$

with $a, b, c, d, e, f \in \mathbb{C}$. Determining the image of these polynomials is considerably more difficult than the bilinear case, but we can deduce the image of many trilinear polynomials.

Lemma 4.1. *The image of $p(X, Y, Z) = XYZ$ is $M_n(\mathbb{C})$.*

Proof. Letting $X = Y = I_n$, we have $p(I_n, I_n, Z) = I_n I_n Z = Z$. Thus, for any matrix $A \in M_n(\mathbb{C})$, we have that $p(I_n, I_n, A) = A$. Thus, $A \in \text{im}_n(p)$ as desired. □

Lemma 4.2. *The image of $p(X, Y, Z) = XYZ - YZX$ is $M_n^0(\mathbb{C})$.*

Proof. Recall that trace is preserved under cyclic permutations. Thus, we have that $\text{tr}(XYZ) = \text{tr}(YZX)$ for all choices of x, y, z. It follows that $\text{tr}(p(X, Y, Z)) = 0$ everywhere. We conclude that $\text{im}(p) \subseteq M_n^0(\mathbb{C})$. To show the other direction, note that

$$p(I_n, Y, Z) = YZ - ZY.$$

We have previously shown that this attains all traceless matrices. Consequently, $\text{im}_n(p) = M_n^0(\mathbb{C})$, as desired. □

The methods above show promise for determining the images of most polynomials; plugging in the identity matrix for one or multiple variables often leads to a polynomial of lower degree whose image is already known. In fact, it seems that a large class of polynomials have an image of everything.

Theorem 4.3. *Let p be a multilinear polynomial of the form*

$$p(X_1, ..., X_d) = \sum_{\sigma \in S_d} \alpha_\sigma \, X_{\sigma(1)} X_{\sigma(2)} \cdots X_{\sigma(d)}.$$

If $\sum \alpha_\sigma \neq 0$, then the image of p is $M_n(\mathbb{C})$.

Proof. The above notation is in reference to the possible permutations of the set of given variables. Note that

$$p(X_1, I_n, \ldots I_n) = \sum_{\sigma \in S_n} \alpha_\sigma X_1 = cX_1$$

for some nonzero $c \in \mathbb{C}$. Thus, for any $A \in M_n(\mathbb{C})$, we have that $p(A/c, I_n, \ldots I_n) = A$. Thus, this implies that $\mathrm{im}_n(p) = M_n(\mathbb{C})$. □

It remains then to determine the images of polynomials in which the sum of coefficients is 0. We can determine the images of some polynomials, as shown below.

Lemma 4.4. *The image of $p(X, Y, Z) = X(YZ - ZY)$ is $M_n(\mathbb{C})$.*

Proof. We know from the bilinear case that $YZ - ZY$ can attain every trace zero matrix. Let $YZ - ZY = B \in M_n^0(\mathbb{C})$ be invertible. Let $A \in M_n(\mathbb{C})$. Then, if $X = AB^{-1}$, then

$$p(X, Y, Z) = AB^{-1}B = A,$$

showing the result. □

The above result relies on factoring. For irreducible trilinear p, the problem becomes more difficult. Consider $p(X, Y, Z) = xyz - zyx$. Our current methods did not allow us to fully determine the image of this polynomial. The full result for all trilinear polynomials, however, was shown by Vitas [13]. We can still obtain clues as to what the image may be.

Lemma 4.5. *Let $p(X, Y, Z) = XYZ - ZYX$. Then,* $\mathrm{im}(p) \supseteq M_n^0(\mathbb{C})$*.*

Proof. Let $x = I$, and note that $p(I, Y, Z) = YZ - ZY$. The result immediately follows. □

While not a full result, showing containment of the traceless matrices can tell us a lot about the full image of a polynomial. In fact, the above example prompts an important question: does plugging in the identity matrix for one variable always give us information about the image of a polynomial?

4.1 More examples and computations

Consider a generic trilinear polynomial defined as before:

$$p(X, Y, Z) = aXYZ + bXZY + cYXZ + dYZX + eZXY + fZYX.$$

We assume that $a + b + c + d + e + f = 0$. Now, if $p(I_n, y, z) \neq 0$ (resp, $p(x, I_n, z)$ and $p(x, y, I_n)$), then $p(I, y, z) = \alpha YZ - \alpha ZY$ for some nonzero α. Thus, $M_n^0(\mathbb{C}) \subseteq \mathrm{im}_n(p)$. It follows that a simplification is not possible if plugging in the identity matrix for any variable always results in the zero polynomial. To determine when exactly this happens, we solve a system of equations below.

$$\begin{aligned} a + b + c + d + e + f &= 0 \\ a + b + c &= 0 \\ d + e + f &= 0 \\ a + b + e &= 0 \\ c + d + f &= 0 \\ a + c + d &= 0 \\ b + e + f &= 0. \end{aligned}$$

This results in the equalities

$$\begin{aligned} a &= f \\ b &= c \\ c &= e = -(a + b). \end{aligned}$$

One example is

$$p(X, Y, Z) = XYZ + XZY - 2YXZ + YZX - 2ZXY + ZYX.$$

Unfortunately, the existence of a counterexample indicates that there may not be a straightforward way to show containment of the traceless matrices for larger matrix sizes. In 2013, Mesyan was able to show this for trilinear polynomials [8]. While some results are known for small d, the question of containment remains open in general. We attempted to apply our methods for the trilinear case to degree 4. This process was difficult and computationally intensive. Moreover, it gives evidence as to why the L'vov-Kaplansky Conjecture is much harder to solve for polynomials of higher degree. We compute conditions for tetralinear polynomials such that plugging in the identity for any variable always produces the zero polynomial i.e

$$p(I_n, Y, Z, W) = p(X, I_n, Z, W) = p(X, Y, I_n, W) = p(X, Y, Z, I_n) = 0.$$

Remark 4.6. Let p be a tetralinear polynomial of the form

$$p(X, Y, Z, W) = a(XYZW) + b(XYWZ) + \cdots + \alpha(WZXY) + \beta(WZYX).$$

If $p(I_n, Y, Z, W) = p(X, I_n, Z, W) = p(X, Y, I_n, W) = p(X, Y, Z, I_n) = 0$, then

$$\begin{aligned}
c + m = e + s = r + \beta = -(u + v) = -(a + b) = -(o + p) \\
a + g = f + t = v + l = -(i + j) = -(\alpha + \beta) = -(c + d) \\
b + h = d + n = p + j = -(k + l) = -(e + f) = -(q + r) \\
g + h = m + n = s + t = -(u + k) = -(i + o) = -(q + \alpha).
\end{aligned}$$

We use the same method as for the trilinear case, but this time we look at all combinations of three variables. This gives the following:

$$\begin{aligned}
a + b + c + m &= 0 & a + g + i + j &= 0 \\
d + e + f + n &= 0 & b + h + k + l &= 0 \\
g + h + i + o &= 0 & c + m + o + p &= 0 \\
j + k + l + p &= 0 & d + n + q + r &= 0 \\
s + t + \alpha + q &= 0 & e + s + u + v &= 0 \\
u + v + \beta + r &= 0 & f + t + \alpha + \beta &= 0 \\
a + b + e + s &= 0 & a + c + d + g &= 0 \\
c + d + f + t &= 0 & b + e + f + h &= 0 \\
g + h + k + u &= 0 & m + n + o + i &= 0 \\
i + j + l + v &= 0 & p + q + r + j &= 0 \\
m + n + q + \alpha &= 0 & s + t + u + k &= 0 \\
o + p + r + \beta &= 0 & v + \alpha + \beta + l &= 0
\end{aligned}$$

Simplifying this system gives the final system shown above.

Increasing the degree of our polynomial increases the complexity of our system of equations. The number of terms increase from 6 to 24, and will increase factorially for larger degree. We do note, however, that the standard polynomial s_4, a known polynomial identity for 2×2 matrices, does satisfy the above system.

5 Examples on 2×2 matrices

For a given multilinear polynomial p, we have that $\text{im}_n(p)$ depends not only on the degree of p, but the size of the matrices, $M_n(\mathbb{C})$, on which we evaluate it.
It was proven by Alexey Kanel-Belov, Sergey Malev, and Louis Rowen that the L'vov-Kaplansky conjecture is true for all multilinear polynomials evaluated on the 2×2 matrices [6]. That is, if p is a multilinear polynomial, we have that $\text{im}_2(p)$ is one of the four possible subspaces: $\{0\}, \mathbb{C}I_2, M_2^0(\mathbb{C})$, or $M_2(\mathbb{C})$.
We provide examples to show that it possible to attain each of these subspaces.
It is simple to find a multilinear polynomial with all of $M_2(\mathbb{C})$ as its image, as a trivial example take $p(X) = X$.
As shown in the bilinear case, for all n, and thus more specifically $n = 2$, we have $\text{im}_n(XY - YX) = M_n^0(\mathbb{C})$.

Definition 5.1. The Amitsur-Levitzki Theorem [1] states that the following polynomial

$$p(X_1, ..., X_4) = \sum_{\sigma \in S_4} \operatorname{sgn}(\sigma) X_{\sigma(1)} \cdots X_{\sigma(4)}$$

is a polynomial identity for $M_2(\mathbb{C})$, called the *standard polynomial on* 2×2 *matrices*. This means that $\operatorname{im}_2(p) = \{0\}$.

Lemma 5.2. *The degree four multilinear polynomial*

$$p(X_1, X_2, X_3, X_4) = [X_1, X_2][X_3, X_4] + [X_4, X_3][X_2, X_1]$$

is a central polynomial for $M_2(\mathbb{C})$*, i.e.,* $\operatorname{im}_2(p) = \mathbb{C}I_2$.

Proof. First, we will show that for all $X_1, X_2, X_3, X_4 \in M_2(\mathbb{C})$, we have $p(X_1, X_2, X_3, X_4) \in \mathbb{C}I_2$. From the bilinear case, we have that $\operatorname{im}_2([X, Y] = XY - YX = M_2^0(\mathbb{C})$. Thus, by the structure of traceless 2×2 matrices, this means that

$$[X_1, X_2] = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \text{ and } [X_3, X_4] = \begin{pmatrix} d & e \\ f & -d \end{pmatrix}$$

for some $a, b, c, d, e, f \in \mathbb{C}$. Then, this means that

$$\begin{aligned} p(X_1, X_2, X_3, X_4) &= \begin{pmatrix} a & b \\ c & -a \end{pmatrix}\begin{pmatrix} d & e \\ f & -d \end{pmatrix} + \begin{pmatrix} d & e \\ f & -d \end{pmatrix}\begin{pmatrix} a & b \\ c & -a \end{pmatrix} \\ &= \begin{pmatrix} ad + bf & ae - bd \\ cd - af & ce + ad \end{pmatrix} + \begin{pmatrix} ad + ce & bd - ae \\ af - cd & bf + ad \end{pmatrix} \\ &= \begin{pmatrix} 2ad + bf + ce & 0 \\ 0 & 2ad + bf + ce \end{pmatrix}. \end{aligned}$$

Therefore, we have that $p(X_1, X_2, X_3, X_4) \in \mathbb{C}I_2$ for all X_1, X_2, X_3, X_4. Hence, $\operatorname{im}_2(p) \subseteq \mathbb{C}I_2$.

Now, let $\lambda \in \mathbb{C}$ be arbitrary, we will show that $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \in \operatorname{im}_2(p)$. We know from the bilinear case that there exist some $X_1, X_2, X_3, X_4 \in M_2(\mathbb{C})$ such that

$$[X_1, X_2] = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} \text{ and } [X_3, X_4] = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}.$$

Then, we have that

$$p(X_1, X_2, X_3, X_4) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}.$$

Thus, from above, we have that $\operatorname{im}_2(p) = \mathbb{C}I_2$. □

The proof of Kanel-Belov, Malev, and Rowen utilizes a ratio of eigenvalues, and inspired by their methods, we will introduce a mapping which takes a matrix to its determinant and trace squared. Using this mapping, we will give a condition which determines when the image of a multilinear polynomial is dense in the 2×2 matrices.

6 Density Approcaches to the L'vov-Kaplansky Conjecture

We introduce mappings and parameterizations to explore the image of a multilinear polynomial p on $M_2(\mathbb{C})$.

Definition 6.1. Define $\Phi : M_2(\mathbb{C}) \to \mathbb{C}^2$ by $\Phi(A) = (\det(A), \mathrm{tr}^2(A))$.

This allows us to move from $M_2(\mathbb{C})$ to $\mathbb{C}^2$. In fact, if Φ can attain a dense space in $\mathbb{C}^2$, then p is dense in $M_2(\mathbb{C})$. Thus, determining information about how Φ behaves in $\mathbb{C}^2$ can give us information of how p behaves in $M_2(\mathbb{C})$. Consider the following proposition, courtesy of Špela Špenko.

Proposition 6.2. *(From [12]) The image of a polynomial p is dense in $M_2(\mathbb{C})$ if and only if $\det(p), \mathrm{tr}(p)$ are algebraically independent.*

Špenko's proof relies uses $\mathrm{tr}(p^2)$ instead of $\det(p)$, but the proposition still holds true in our case. Now, we introduce a parameterization, defined as follows:

Definition 6.3. Let $X, Y \in M_2(\mathbb{C})^d$. Define a parametric equation $\gamma(z) : \mathbb{C} \to M_2(\mathbb{C})^d$ as

$$\gamma(z) = X + z(Y - X).$$

Now, we can combine our parameterization with our mapping in order to gain insight in $\mathrm{im}_2(p)$ for multlinear p. We additionally introduce a scaling parameter w.

Definition 6.4. Let $g(w, z) = \Phi(p(w \cdot \gamma(z)))$.

This prompts the following question.

Question 6.5. *Suppose $p(X_1, \ldots, X_d)$ is a multilinear polynomial. Assume that there exist $X, Y \in M_2(\mathbb{C})^d$ such that $\Phi(p(X))$ and $\Phi(p(Y))$ are linearly independent. Is $\{g(w, z) : w, z \in \mathbb{C}\}$ dense in $\mathbb{C}^2$?*

Here, note that for multilinear p, $p(wx_1, \ldots, wx_d) = w^d p(x_1, \ldots, x_d)$. Answering this question will involve properties of the Jacobian matrix, defined as follows:

Definition 6.6. Suppose $f : \mathbb{C}^m \to \mathbb{C}^n$ is differentiable at x_0. The *Jacobian Matrix* of f is the $n \times m$ matrix of functions denoted by

$$J_f(x_0) := \left(\frac{\partial f_i}{\partial x_j}(x_0) \right)_{i,j=1}^{m,n}.$$

Note that $Df(x_0)[h] = J_f(x_0)h$, where the product on the right hand side is matrix multiplication against a column vector. An $n \times m$ matrix is a linear map from $\mathbb{C}^m$ to $\mathbb{C}^n$ when using multiplication on the right by column vectors. Since $f : \mathbb{C}^m \to \mathbb{C}^n$, its derivative at a point should have the same action, which is how we've built our Jacobian matrix.

Recall that $g(w, z)$ is a polynomial map from $\mathbb{C}^2$ to $\mathbb{C}^2$. Thus, its partial derivatives are polynomials as well (the Jacobian matrix is a 2×2 matrix of polynomials in w and z), and consequently the determinant of the Jacobian matrix is a commutative polynomial in w and z.

So, does the linear independence of the points X, Y help us show the derivative is invertible?

Let p be a multilinear polynomial. Assume $\Phi(p(X))$ and $\Phi(p(Y))$ are not colinear. Note that the homogeneity of p implies that g is homogeneous in w. If $p(wA) = w^d p(A)$, then

$$g(z, w) = w^{2d} g(z, 1) = w^{2d} \Phi(p(\gamma(z))).$$

Here d is the homogeneity degree of p, while 2 comes from the 2×2 matrix case.

Observe that

$$\frac{\partial g}{\partial w} = d w^{d-1} g(z, 1).$$

Also note that $\frac{\partial g}{\partial z}(z_0, 1)$ and $\frac{\partial g}{\partial w}(z_0, 1)$ are parallel vectors if and only if $\frac{\partial g}{\partial z}(z_0, 1) = \kappa(z_0) g(z_0, 1)$ for some polynomial κ. That is, J_g is invertible at a point if and only if the z partial derivative is not equal to the function value at some point.

Question 6.7. *Let q be a polynomial. If $q'(z_0) = \kappa(z_0) q(z_0)$ for all $z_0 \in \mathbb{C}$, then is q effectively linear?*

An example: $q_1(z) = (z^2, 4z^2)$. This is not linear in z, but its image is a 1-dimensional subspace of $\mathbb{C}^2$.

Lemma 6.8. *Suppose $q : \mathbb{C}^n \to \mathbb{C}^n$ is a polynomial mapping. If there exists a point $x_0 \in \mathbb{C}^n$ such that $Dq(x_0)$ is invertible, then $\operatorname{im}(q)$ is a Zariski-dense subset of $\mathbb{C}^n$.*

Proof. Let $q = (q_1, \ldots, q_n)$, where each $q_i \in \mathbb{C}[x_1, \ldots, x_n]$. Suppose $Dq(x_0)$ is invertible. Next, assume by way of contradiction that the image of q is not Zariski dense. It follows that there exists a nonzero polynomial r such that $r(q_1(x), \ldots, q_n(x)) = 0$.

The Inverse Function Theorem says that there exists a neighborhood about x_0 on which q is invertible and its local inverse is differentiable. Thus, q is an open mapping near x_0, so that it sends the open neighborhood about x_0 to an open set. Namely, the image of q contains an open neighborhood about $p(x_0)$. It follows that r must vanish on this open neighborhood, implying that r is the zero polynomial, a contradiction. Therefore, the image of q is Zariski dense in $\mathbb{C}^n$. □

Remark 6.9. An example of a Zariski-dense set is the domain of the rational mapping

$$(z, w) \mapsto (z^{-1}(1 - zw)^{-1},\ (1 + 2z + w + w^2 - z^2)^{-2}).$$

This rational function is defined at any point $(z, w) \in \mathbb{C}^2$ except for the points that make either z, $1 - zw$, or $1 + 2z + w + w^2 - z^2$ equal to zero.

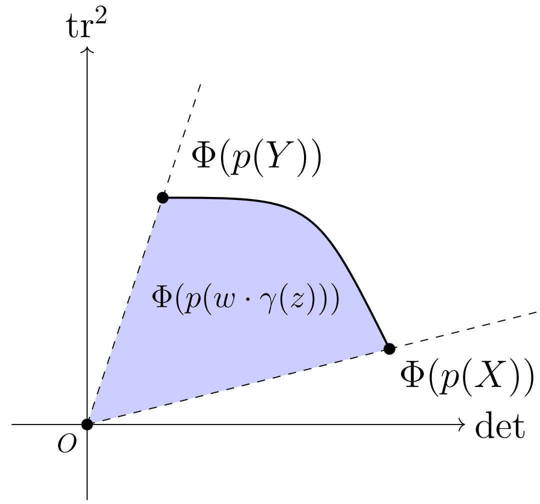

Figure 1: A depiction of our mapping into $\mathbb{C}^2$. Observe that scaling our polynomial allows us to 'fill in' our space.

Theorem 6.10. *Let p be a multilinear polynomial of degree d.*

If there exist points $X, Y \in M_2(\mathbb{C})^d$ such that $\Phi(p(X))$ and $\Phi(p(Y))$ are not linearly dependent, then $\mathrm{im}_2(p)$ is a Euclidean-dense subset of $M_2(\mathbb{C})$.

Proof. Suppose there exist $X, Y \in M_2(\mathbb{C})^d$ such that that $\Phi(p(X))$ and $\Phi(p(Y))$ are linearly independent vectors in $\mathbb{C}^2$.
Define $\gamma(z)$ and $g(z, w)$ as before and let $f(z) = (\Phi \circ p \circ \gamma)(z)$. We note that $f(z) = (f_1(z), f_2(z))$ for some polynomials f_1 and f_2. Now, we consider the Jacobian matrix of g at $(z, 1)$. We have

$$J_g(z,1) = \begin{pmatrix} f_1'(z) & f_2'(z) \\ df_1(z) & df_2(z) \end{pmatrix}.$$

Suppose there exists some $z_0 \in \mathbb{C}$ such that $J_g(z_0, 1)$ is invertible. So, g is a polynomial mapping from $\mathbb{C}^2$ to $\mathbb{C}^2$ such that $Dg(z_0)$ is invertible. This implies, by Lemma 6.8 that $\mathrm{im}(g)$ is a Zariski-dense subset of $\mathbb{C}^2$.
Now, observe that for all $(z, w) \in \mathbb{C}^2$, we have that $g(z, w) = \Phi(p(w \cdot \gamma(z)))$. Thus, $g(z, w) \in \mathrm{im}(\Phi \circ p)$, and hence $\mathrm{im}(g) \subseteq \mathrm{im}(\Phi \circ p)$. Therefore, because the image of g is a dense subset of $\mathbb{C}^2$, we must have that the image of $\Phi \circ p$ is a dense subset of $\mathbb{C}^2$.
By Proposition 5.2 in [12], the density of $\Phi \circ p$, and thus density of the trace and determinant, implies that $\mathrm{im}_2(p)$ is a dense subset of $M_2(\mathbb{C})$. Therefore, if $J_g(z_0, 1)$ is invertible for any $z_0 \in \mathbb{C}$, we have that the image of p is a dense subset of $M_2(\mathbb{C})$.
We now consider the case when $J_g(z, 1)$ is not invertible for all $z \in \mathbb{C}$. This

means that for all $z \in \mathbb{C}$, we have $\det(J_g(z,1)) = 0$. Therefore, by the equation for the determinant of a 2×2 matrix, we have

$$d(f_1'(z)f_2(z) - f_1(z)f_2'(z)) = 0. \tag{1}$$

If $f_1(z) = f_2(z) = 0$ for all $z \in \mathbb{C}$, then this would contradict our assumption that $f(0) = \Phi(p(X))$ and $f(1) = \Phi(p(Y))$ are linearly independent.
So, assume that $f_1(z) \not\equiv 0$, using (1), we have

$$\frac{d}{dz}\left(\frac{f_2(z)}{f_1(z)}\right) = \frac{f_1'(z)f_2(z) - f_1(z)f_2'(z)}{(f_1(z))^2} = 0.$$

This implies that the ratio between $f_2(z)$ and $f_1(z)$ is constant, and thus $f_1(z) = cf_2(z)$ for some $c \in \mathbb{C}$. Therefore, we have that $\mathrm{im}(f)$ is contained in a proper subspace of $\mathbb{C}^2$. This contradicts our assumption that $f(0)$ and $f(1)$ are linearly independent.
Hence, given that $\Phi(p(X))$ and $\Phi(p(Y))$ are linearly independent, we must have that $\mathrm{im}_2(p)$ is a dense subset of $M_2(\mathbb{C})$. □

The image of our mapping in $\mathbb{C}^2$ can either be a dense subset of the full space or a dense subset of a line through the origin. We know from established results that the only possibilities that this line can be are $(z, 4z)$ (the scalar matrices) or $(z, 0)$ (the traceless matrices). However, our proof does not rule out other possibilities.

Question 6.11. *Can we rule out any other subspaces as the images of* $\Phi \circ p$*?*

6.1 Generalizing to Larger Matrix Sizes

One main motivation for pursuing the density approach was the hope of generalizing our 2×2 approach to larger matrix sizes.

Definition 6.12. A *cone* is a set that is closed under scalar multiplication.

Example 6.13. The subspace $\{(z, 4z) : z \in \mathbb{C}\}$ is a cone in $\mathbb{C}^2$.

Remark 6.14. The 2×2 argument uses the fact that any one dimensional cone must be a line or a set of lines through the origin. Our proof shows that the image of Φ cannot be set of lines; the image is either the full space or a line through the origin (up to density). Since a line through the origin is a subspace, the image of Φ must be a subspace in the 2×2 case. However, when generalizing to $n \times n$ matrices, we can no longer guarantee that an $(n-1)$-dimensional subset is a subspace. We present our work for the 3×3 case as an example. We introduce similar mappings, defined as follows.

Definition 6.15. Define $\Phi_3 : M_3(\mathbb{C}) \to \mathbb{C}^3$ by $\Phi(A) = (\det^2(A), \mathrm{tr}^6(A), \mathrm{tr}^3(A^2))$.

Now, we introduce a parameterization, defined as follows:

Definition 6.16. Let $X, Y \in M_3(\mathbb{C})^d$. Define a parametric equation $\gamma_3(z_1, z_2)$: $\mathbb{C}^2 \to M_3(\mathbb{C})^d$ as

$$\gamma(z_1, z_2) = X + z_1(Y - X) + z_2(Z - X).$$

Definition 6.17. Let $g_3(w, z_1, z_2) = \Phi(p(w \cdot \gamma_3(z_1, z_2)))$.

Our attempted proof follows closely to the 2×2 case, but fails when we assume noninvertibility everywhere.

Question 6.18. *Let $g : \mathbb{C}^2 \to \mathbb{C}^3$ be a polynomial. If for all $z_0 \in \mathbb{C}^2$, $p(z_0) \in \operatorname{span}\{\frac{\partial g}{\partial x_i}\}$, then is g is contained in a 2-dimensional subspace?*

Say g has two input variables z_1, z_2, and output

$$g(z_1, z_2) = \langle g_1(z_1, z_2), g_2(z_1, z_2), g_3(z_1, z_2)\rangle$$

Consider the following matrix J_{g_3}, defined as follows:

$$J_{g_3} = \begin{pmatrix} \frac{\partial g_1}{\partial z_1} & \frac{\partial g_2}{\partial z_1} & \frac{\partial g_3}{\partial z_1} \\ \frac{\partial g_1}{\partial z_2} & \frac{\partial g_2}{\partial z_2} & \frac{\partial g_3}{\partial z_2} \\ g_1 & g_2 & g_3 \end{pmatrix}.$$

If the hypothesis is true, then J_{g_3} is noninvertible and thus $\det(J_{g_3}) = 0$. Computing directly, it can be shown that

$$\begin{aligned} g_1(\frac{\partial p_2}{\partial x_1}\frac{\partial g_3}{\partial x_2} - \frac{\partial g_2}{\partial x_2}\frac{\partial g_3}{\partial x_1}) + g_2(\frac{\partial g_1}{\partial x_2}\frac{\partial g_3}{\partial x_1} - \frac{\partial g_1}{\partial x_1}\frac{\partial g_3}{\partial x_2}) \\ + g_3(\frac{\partial g_1}{\partial x_1}\frac{\partial g_2}{\partial x_2} - \frac{\partial g_2}{\partial x_1}\frac{\partial p_1}{\partial g_2}) = 0 \\ g_3 = \frac{g_1(\frac{\partial g_2}{\partial x_1}\frac{\partial g_3}{\partial x_2} - \frac{\partial g_2}{\partial x_2}\frac{\partial g_3}{\partial x_1}) + g_2(\frac{\partial g_1}{\partial x_2}\frac{\partial g_3}{\partial x_1} - \frac{\partial g_1}{\partial x_1}\frac{\partial g_3}{\partial x_2})}{\frac{\partial g_2}{\partial x_1}\frac{\partial g_1}{\partial x_2} - \frac{\partial g_1}{\partial x_1}\frac{\partial g_2}{\partial x_2}} \end{aligned}$$

Rearranging gives us a formula for p_3 in terms of p_1 and p_2. However, this is not enough information to guarantee a subspace.

Remark 6.19. We only know that $\operatorname{im}_3(p)$ is a two-dimensional cone in $\mathbb{C}^3$. While we hoped to show that this necessarily is a subspace, our methods failed to exclude potential counterexamples, such as the standard geometric interpretation of a cone.

Example 6.20. Consider $f(z_1, z_2) = (z_1^2, z_2^2, z_1 z_2)$. The image of f is not a plane in $\mathbb{C}^2$ but does satisfy the definition of a cone.

Remark 6.21. For a general n, our Φ_n is written in terms of

$$\det(A), \operatorname{tr}(A), ..., \operatorname{tr}(A^{n-1})$$

in accordance with Špela Špenko's result. As in Φ_3, additional exponents are added to ensure that scaling is consistent across each term.

Question 6.22. *Can we modify our 2×2 proof to eliminate non-subspace counter examples?*

7 Factoring Multilinear Polynomials

Seeking to reduce the complexity of our multilinear polynomials in any way possible, we turn our attention to factoring multilinear polynomials. If a multilinear polynomial p is reducible, does this tell us anything about its image? Our goal is to determine if there are additional conditions that we can impose on a multilinear polynomial in order to verify that it satisfies the Density Dimension-Free L'vov-Kaplansky Conjecture. One important result that we proved was that multilinearity is preserved by factoring.

Definition 7.1. Let w be a term in p with a nonzero coefficient. Define $d_{x_i}(p) := \max\{\deg_{x_i}(w) : w \in p\}$. If $p = 0$, we say $d_{x_i} = -\infty$.

We note that for multilinear polynomials p_1, p_2,

$$d_{x_i}(p_1 p_2) = d_{x_i}(p_1) + d_{x_i}(p_2).$$

Furthermore,

$$d_{x_i}(p_1 + p_2) \leq \max\{d_{x_i}(p_1), d_{x_i}(p_2)\}.$$

Theorem 7.2. *If p is a multilinear polynomial and it factors as $p = p_1 p_2 ... p_k$ such that none of the p_i is a constant polynomial, then each p_i is multilinear.*

Proof. Without loss of generality, suppose $p = p_1 p_2$. Let $p_1 = p(X_1, ..., X_m)$ and $p_2 = p(Y_1, ..., Y_n)$. We cannot assume that p_1 and p_2 are multilinear, so we consider each noncommutative polynomial to be the sum of words or monomials.

For every variable X_k in p_1 (resp. p_2), $d_{X_k}(p_1) \geq 1$ Suppose by way of contradiction that p_1, p_2 share a common variable X_j. Then, $d_{X_j}(p) \geq 2$ since

$$d_{X_j}(p_1) + d_{X_j}(p_2) = d_{x_j}(p)$$

by the properties of our X_j-degree valuation. However, we know that $d_{X_i} = 1$. for every variable X_i in p since p is multilinear. Thus p_1 and p_2 are polynomials in distinct variables.

Now, without loss of generality we will show that p_1 is multilinear; it suffices to show that p_1 is linear in x_1. By the multilinearity of p, we have that

$$\begin{aligned} p(X_1 + cZ, ..., X_m, Y_1, ..., Y_n) &= p(X_1, ..., X_m, Y_1, ..., Y_n) + cp(Z, ..., X_m, Y_1, ..., Y_n) \\ &= p_1(X_1, ..., X_m)p_2(Y_1, ..., Y_n) + cp_1(Z, ..., X_m)p_2(Y_1, ..., Y_n) \\ &= (p_1(X_1, ..., X_m) + cp_1(Z, ..., X_m))p_2(Y_1, ..., Y_n) \end{aligned}$$

Because we also know that

$$p(X_1 + cZ, ..., X_m, Y_1, ..., Y_n) = p_1(X_1 + cZ, ..., X_m)p_2(Y_1, ..., Y_n),$$

this implies that

$$p_1(X_1 + cZ, ..., X_m)p_2(Y_1, ..., Y_n) = (p_1(X_1, ..., X_m) + cp_1(X, ..., X_m))p_2(Y_1, ..., Y_n).$$

Consequently,

$$\big(p_1(X_1 + cZ, ..., X_m) - (p_1(X_1, ..., X_m) + cp_1(Z, ..., X_m))\big)p_2(Y_1, ..., Y_n) = 0.$$

However, p_2 is nonzero, and $\mathbb{C}\langle X, Y\rangle$ is a domain [3], thus p_1 is multilinear, and similarly p_2 is multilinear.
Hence, we have shown that if $p = p_1p_2$ for some nonconstant polynomials p_1, p_2, then p_1 and p_2 are multilinear in distinct variables. □

This result is significant in that it allows us to more efficiently study multilinear polynomials of higher degree by factoring them into multilinear polynomials of smaller degree, for which we may already have results. Inspired by this result, we prove several related results about the factorization of polynomials. To do so, we utilize two results from Špela Špenko, which we have termed the "Density Domino Effect". These are Lemma 5.3 in [12]. This first result states that if we have density in the traceless matrices for one matrix size, this density in the traceless matrices persists for all larger matrix sizes.

Lemma 7.3. *If p is a multilinear polynomial, and $\mathrm{im}_{n-1}(p) \cap M^0_{n-1}(\mathbb{C})$ is dense in $M^0_{n-1}(\mathbb{C})$, then $\mathrm{im}_n(p) \cap M^0_n(\mathbb{C})$ is dense in $M^0_n(\mathbb{C})$.*

This implies that all multilinear polynomials that are not central polynomials or polynomial identities for 2×2 matrices must attain density in the traceless matrices, $M^0_n(\mathbb{C})$, for all $n \geq 2$. Additionally, we have

Remark 7.4. If p is a multilinear polynomial, $\mathrm{im}_n(p) \cap M^0_n(\mathbb{C})$ is dense in $M^0_n(\mathbb{C})$, and $\mathrm{im}_n(p) \nsubseteq M^0_n(\mathbb{C})$, then $\mathrm{im}_n(p)$ is dense in $M_n(\mathbb{C})$.

Thus, if a multilinear polynomial with density in the traceless matrices attains even one other matrix with trace nonzero, then the image is dense in the full matrix algebra. The following is a consequence of the two previous results and the fact that once a multilinear polynomial attains one trace non-zero matrix, this persists for larger matrix sizes. Thus,

Corollary 7.5. *If p is a multilinear polynomial and $\mathrm{im}_{n-1}(p)$ is dense in $M_{n-1}(\mathbb{C})$, then $\mathrm{im}_n(p)$ is dense in $M_n(\mathbb{C})$.*

Before continuing the discussion, we introduce a well-known Lemma, e.g. [9]:

Lemma 7.6. *If $p \in \mathbb{C}\langle x_1, \ldots, x_d\rangle$ then either there exists $n \in \mathbb{N}$ and a matrix tuple $X = (X_1, \ldots, X_d)$ so that $p(X)$ is invertible, or p is the zero polynomial.*

Proof. Suppose p is not the zero polynomial. Posner's Theorem implies that the $n \times n$ generic matrix algebra is contained in a skew field, UD_n. In particular, an evaluation of our polynomial on generic matrices is an element of UD_n. Hence, each evaluation is either zero, or it is invertible within UD_n.

In the former setting, it follows that the polynomial p also evaluates to zero identically on $n \times n$ matrices, hence, by our assumption that p is not the zero polynomial there must be some matrix size n sufficiently large so that p evaluates to an invertible element of UD_n. As a nonzero element of UD_n, the determinant of the evaluation is an invertible element of some rational field over $\mathbb{C}$ (rational over commutative indeterminants that are the entries of the generic matrices). It follows that there exists some evaluation of this (commutative) rational function that is nonzero. Since the indeterminants of the rational function are the entries of the generic matrices, we conclude that there exist matrices $X = (X_1, \ldots, X_d)$ so that $p(X)$ is invertible. $\square$

In terms of the Density Dimension-Free L'vov-Kaplansky Conjecture, showing that multilinear polynomials attain density in the traceless matrices for a given size is equivalent to proving the conjecture. We will now detail two results that we proved about factoring polynomials using the "Density Domino Effect." In our proof, we will use the Frobenius matrix norm.

Theorem 7.7. *Let p be a multilinear polynomial that factors;*

$$p = p_1 \ldots p_k.$$

If there exists an i, n such that $\mathrm{im}_n(p_i)$ is dense in $M_n(\mathbb{C})$, then $\mathrm{im}_m(p)$ is dense in $M_m(\mathbb{C})$ for some $m \geq n$.

Proof. It suffices to check the case where $p = p_1 p_2 p_3$ and p_2 is dense in $M_n(\mathbb{C})$. Let $\varepsilon > 0$. We know that p_1, p_3 are not polynomial identities over $M_m(\mathbb{C})$ for some m by the Amitsur-Levitzki theorem [1]; assume $m > n$. We then know that $\mathrm{im}_m(p_1), \mathrm{im}_m(p_3)$ must each attain an invertible matrix in their image by Lemma 7.6. Let A_1, A_3 be invertible matrices in $\mathrm{im}_m(p_1), \mathrm{im}_m(p_3)$, respectively. Suppose $X \in M_m(\mathbb{C})$. Then, by Remark 7.5, there exists a $B \in \mathrm{im}_m(p_2)$ such that $||B - X|| < \varepsilon/2$. Additionally, we know that there exists a matrix $Y \in \mathrm{im}_m(p_2)$ such that $||Y - A_1^{-1} B A_3^{-1}|| < \tilde{\varepsilon}$, where

$$\tilde{\varepsilon} < \frac{\varepsilon}{2(||A_1||)(||A_3||)}.$$

We have that $\|A_1\|, \|A_3\| > 0$ because A_1, A_3 are invertible. Denote $D = Y - A_1^{-1} B A_3^{-1}$. Thus, the image of p contains

$$A_1(A_1^{-1} B A_3^{-1} + D) A_3 = B + A_1 D A_3.$$

Now, we know that $||B - X|| < \varepsilon/2$ and $||A_1 D A_3|| < \varepsilon/2$. Thus, by the triangle inequality, we have that

$$||B + A_1 D A_3 - X|| < \varepsilon.$$

We conclude that the image of p is dense in $M_m(\mathbb{C})$ for some m, as desired. $\square$

This result is useful because if a multilinear polynomial is reducible, then we only need to know that one of its factors attains density in the full matrix algebra. However, what can we say when none of the factors attain density in the full space? Our next result implies that density in the traceless matrices of two adjacent factors would allow us to appeal to Theorem 7.7.

Theorem 7.8. *If p_1 and p_2 are two polynomials such that $\operatorname{im}_n p_1 \cap M_n^0(\mathbb{C})$ and $\operatorname{im}_n p_2 \cap M_n^0(\mathbb{C})$ are dense in $M_n^0(\mathbb{C})$, then $\operatorname{im}_n p_1 p_2$ is dense in $M_n(\mathbb{C})$.*

Proof. Let $A \in M_n(\mathbb{C})$ and $\varepsilon > 0$ be arbitrary. We will show there exists some $Z \in \operatorname{im}_n(p_1 p_2)$ such that $\|Z - A\| < \varepsilon$.
Because p_1 and p_2 are multilinear, using the permutation notation, it is clear that $\mathbf{0} \in \operatorname{im}_n(p_1), \operatorname{im}_n(p_2)$. Thus, $\mathbf{0} \in \operatorname{im}_n(p_1 p_2)$. Therefore, we can assume that A is not the zero matrix. By de Saguin-Pazzis [10] Theorem 3 and Proposition 12, there exist some trace zero matrices $B, C \in M_n^0(\mathbb{C})$ such that $A = BC$. Because A is not the zero matrix, neither B nor C can be the zero matrix. Therefore, $\|B\|, \|C\| > 0$. Let $M = \max\{\|B\|, \|C\|, 1\}$.
Now, let $\varepsilon' = \min\{\varepsilon, 1\}$. Because the image of p_1 is dense in $M_n^0(\mathbb{C})$, there exists some $X \in \operatorname{im}_n(p_1)$ such that $\|X - B\| < \frac{\varepsilon'}{3M}$. Similarly, there exists some $Y \in \operatorname{im}_n(p_2)$ such that $\|Y - C\| < \frac{\varepsilon'}{3M}$. We have $XY \in \operatorname{im}_n(p_1 p_2)$, and observe that using the triangle inequality and sub-multiplicativity of the matrix norm we have

$$\begin{aligned}
\|XY - A\| &= \|XY - BC\| \\
&= \|(B + (X - B))(C + (Y - C)) - BC\| \\
&= \|B(Y - C) + (X - B)C + (X - B)(Y - C)\| \\
&\leq \|B\|\|Y - C\| + \|C\|\|X - B\| + \|X - B\|\|Y - C\| \\
&< \|B\|\frac{\varepsilon'}{3M} + \|C\|\frac{\varepsilon'}{3M} + \frac{(\varepsilon')^2}{9M^2}.
\end{aligned}$$

Because $M \geq \|B\|, \|C\|$, we have $\|B\|\frac{\varepsilon'}{3M} \leq \frac{\varepsilon'}{3}$ and $\|C\|\frac{\varepsilon'}{3M} \leq \frac{\varepsilon'}{3}$. Also, because $\varepsilon' \leq 1$, we have $(\varepsilon')^2 \leq \varepsilon'$. Additionally, because $M \geq 1$, we have $M^2 \geq 1$, thus $9M^2 \geq 3$. Hence, we have $\frac{(\varepsilon')^2}{9M^2} \leq \frac{\varepsilon'}{3}$. Therefore, this implies, using the inequality from above, that

$$\begin{aligned}
\|XY - A\| &< \frac{\varepsilon'}{3} + \frac{\varepsilon'}{3} + \frac{\varepsilon'}{3} \\
&= \varepsilon.
\end{aligned}$$

Finally, since $\varepsilon' \leq \varepsilon$, this means that $\|XY - A\| < \varepsilon$.
Therefore, for an arbitrary $\varepsilon > 0$ and $A \in M_n(\mathbb{C})$, there exists some $Z = XY \in \operatorname{im}_n(p_1 p_2)$ such that $\|Z - A\| < \varepsilon$. Hence, the image of $p_1 p_2$ is dense in $M_n(\mathbb{C})$. ☐

Therefore, we have shown that reducibility of a multilinear polynomial can be extremely useful in determining its image on $M_n(\mathbb{C})$. However, it remains open what can be said about irreducible polynomials.

8 Future Directions

There are several directions to take following the density approach. Generally, if we can show that every nonzero multilinear polynomial eventually has density in $M_n^0(\mathbb{C})$, then only irreducible polynomials can fail to attain density in $M_n(\mathbb{C})$. To do this we would need to investigate if irreducible polynomials have a specific form. Additionally, by looking at images of standard polynomials, central polynomials, and polynomial identities, we could gain further insight to reducible and irreducible polynomials forms. Looking at these polynomials could answer the question of what operations on a multilinear polynomial preserve density in $M_n^0(\mathbb{C})$. Another open question is at what dimension does the "density domino effect" from [12] start for a given multilinear polynomial? Also, we still hope to find a density proof that generalizes, since the proof we provided for the 2×2 case in Section 6 revealed itself not to be. If we know what dimension k the domino effect starts, we simply need to find something that generalizes up to k. A different approach to the problem could include utilizing differential geometry as opposed to an analytical one. One idea to do this is looking at the curvature of images, possibly using the Laplacian Equation.

9 Acknowledgments

We would like to thank the James Madison University Department of Mathematics and Statistics, Dr. Méric Augat, and the National Science Foundation for their support through grants DMS-2349593 and DMS-2522763. We also thank our home institutions of Bryn Mawr College, Longwood University, and Bucknell University.

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