Non-local approximation properties of $\varphi(x) = x^{-1}\ln(1+x^2)$

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Introduction

This project concerns approximation properties of the set

$$S(\varphi, X) := \left\{ \sum_{j=1}^{N} a_j \varphi(x - x_j) : a_j \in \mathbb{R}, x_j \in X \right\},$$

where X is a scattered sequence and $\varphi(x) = x^{-1}\ln(1+x^2)$. Similar approximation sets are commonly used in interpolation problems and are especially helpful due to their Fourier representation. For our work, we will work to prove the following theorem.

Main Result

Suppose $f \in C[a,b]$. For any $\varepsilon > 0$, there exists $s \in S$, such that

$$\|f - s\|_{L_\infty} < \varepsilon.$$

We begin with the Taylor Series for

$$\varphi(x-y) = (x-y)^{-1}\ln(1+(x-y)^2)$$

which yields

$$\varphi(x-y) := \ln|y| \sum_{j=1}^{\infty} \frac{A_j(x)}{y^j} + \sum_{k=2}^{\infty} \frac{B_k(x)}{y^k}.$$

for some polynomials $A_j(x)$ and $B_k(x)$. Using methods from linear algebra, we then collect A_j. Our interest was spurred by approximation theoretic results, namely those found in [1], [2], and [3].

Cauchy Product

The Cauchy Product is the blending of two power series. Let

$$\sum_{n=0}^{\infty} a_n \text{ and } \sum_{n=0}^{\infty} b_n$$

be two series. The Cauchy Product of these two series is defined as the sum

$$\sum_{n=1}^{\infty} a_n \text{ where } c_n = \sum_{k=0}^{n} a_k b_{n-k} \text{ for all } n \in 0, 1, 2, \ldots$$

Vandermonde Matrix

The Vandermonde Matrix is a matrix in which each element increases in a geometric pattern by row or column.

$$V = \begin{bmatrix} 1 & x_1 & x_1^2 & \ldots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \ldots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_m & x_m^2 & \ldots & x_m^{n-1} \end{bmatrix}$$

The determinant of a square Vandermonde matrix can be expressed as

$$\det(V) = \prod_{1 \le i < j \le n} (x_j - x_i).$$

$\varphi(x) = x^{-1}\ln(1+x^2)$

$$\begin{aligned} \frac{\partial}{\partial y}(\varphi(x-y)) &= \frac{-2(x-y)}{1+(x-y)^2} \\ &= \sum_{n=0}^{\infty} \frac{2A_{k-1}(x)}{y^k} + \sum_{k=2}^{\infty} \frac{-2xA_{k-2}(x)}{y^k} \\ &= \frac{2A_0(x)}{y} + \sum_{k=2}^{\infty} \frac{2A_{k-1}(x) - 2xA_{k-2}(x)}{y^k} \end{aligned}$$

And from [1] we know that $A_n(x) = (n+1)x^n +$ lower order terms
Note: $A_0 = 1$ So we get that:

$$\frac{\partial}{\partial y}(\varphi(x-y)) = \frac{2}{y} + \sum_{k=2}^{\infty} \frac{B_k(x)}{y^k}$$

We can then write the sum of the series

$$\sum_{j=1}^{\infty} \frac{A_j(x)}{y^j} + \sum_{k=2}^{\infty} \frac{B_k(x)}{y^k}$$

With

$$A_j = \ln|y| \sum_{j=1}^{\infty} \frac{-2x^{j-1}}{y^j} \text{ and } B_k = \left[\sum_{j=1}^{\infty} \frac{x^{j-1}}{y^j}\right] \left[\sum_{k=1}^{\infty} \left(\frac{B_{k+1}(x)}{k}\right) \frac{1}{y^k}\right]$$

Simplifying A_j, we get that

$$A_j = -2x^{j-1}, j \ge 1.$$

The Cauchy Product is used to simplify B_k.

$$\sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \frac{x^{j-1}B_{k+1}(x)}{k} \frac{1}{y^{j+k}}.$$

When $m = j + k$ and $j = m - k$, we then use substitution and get

$$\sum_{m=2}^{\infty} \left(\sum_{k=1}^{m-1} \frac{x^{m-k-1}B_{k+1}(x)}{k} \right) \frac{1}{y^m}$$

where $2 \le m \le \infty$ and $1 \le k \le m-1$

So,

$$C_m(x) = \sum_{k=1}^{m-1} \frac{x^{m-k-1}2x^k}{k} = \left(\sum_{k=1}^{m-1} \frac{2}{k} \right) x^{m-1}$$

Approximations and Differences in Functions

4th degree approximation and difference in polynomials.

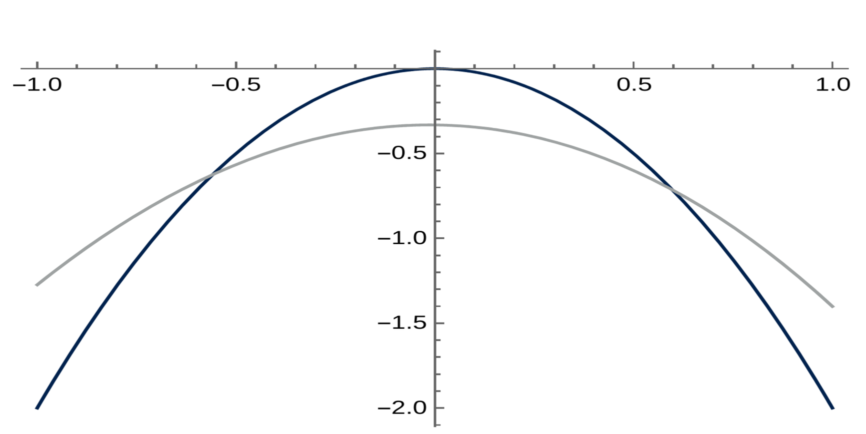


Figure 1. $\varphi(x)$ 4x4 Approximation

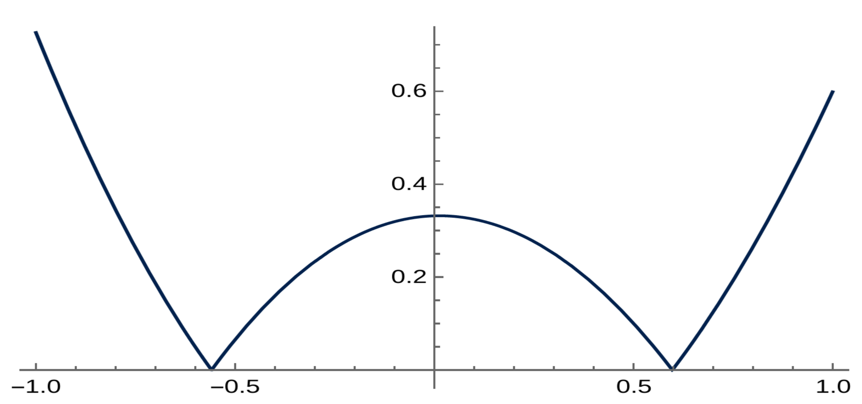
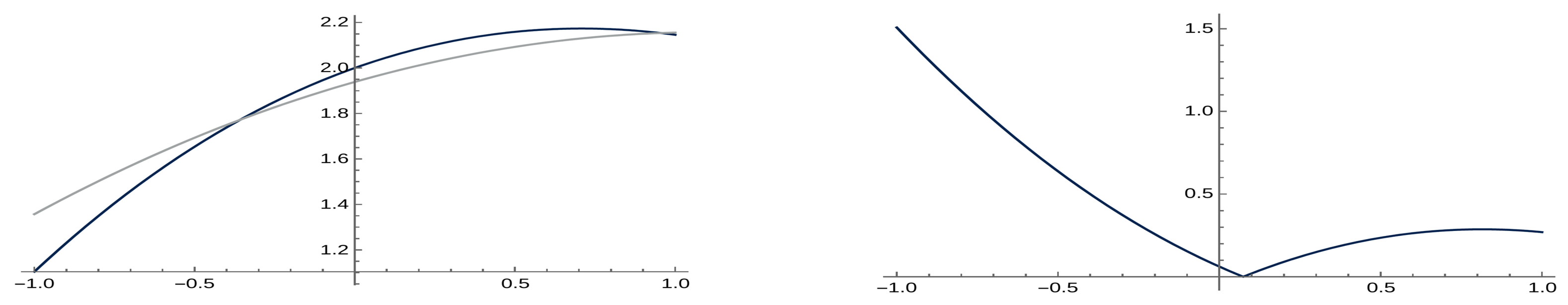


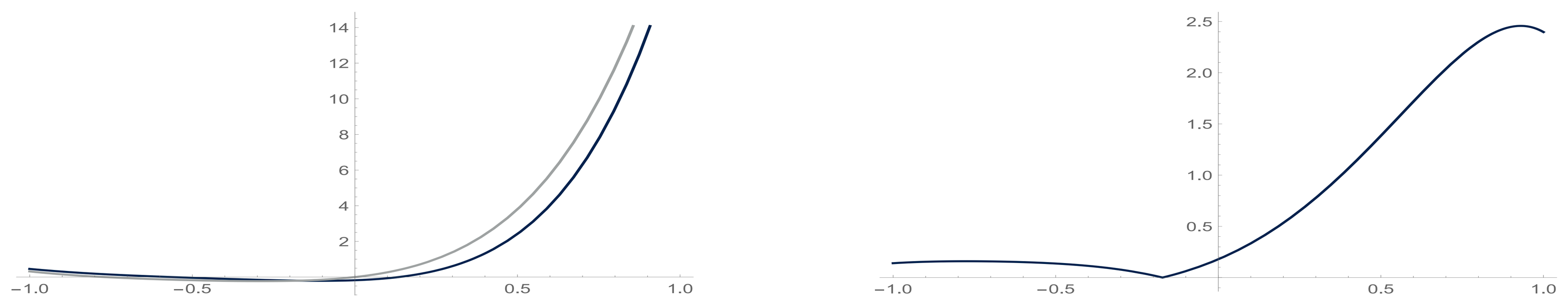
Figure 2. $\varphi(x)$ 4x4 Difference

Examples

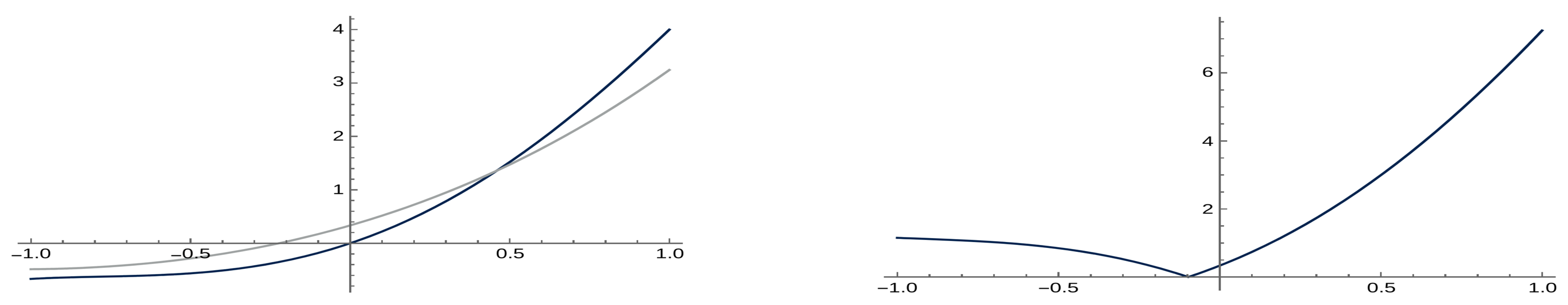
$e^{1/2x} + \cos x$



$e^{3x} \sin 2x$



$e^{x} \sin(2 \tan x)$



Conclusion and Further Work

As shown, we know that our polynomials of $A_j(x)$ form a basis for Π, therefore $\varphi(x) = x^{-1}\ln(1+x^2)$ can be used for its approximating properties. Furthermore, it is true that the same approximation scheme is applicable for the arbitrary values of q and r in

$$(1+x^q)^r$$

where q is any natural number and r is any real number that is not a natural number.

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[2] J.Ledford, On the density of scattered translates of the general multiquadratic in C([a,b]). New York J. Math. **20** (2014), 145–151.

[3] M.J.D. Powell,Univariate multiquadric interpolation: Reproduction of linear polynomials, in *Multivariate Approximation and Interpolation (Duisberg 1989)*, Internat. Ser. Numer. Math. **94**, 227-240, Birkhäuser, Basel, 1990.