Pattern Avoidance in Cyclic Permutations

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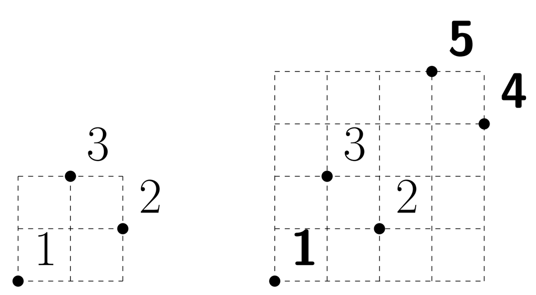
Abstract

Pattern avoidance in permutations is a well-studied field of enumerative combinatorics. We will discuss the classical version for linear permutations and then introduce a recent variant for cyclic permutations. Finally, we will present our new results counting cyclic avoidance sets for pairs of length 4 patterns, and give an example of how those results arise from counting arguments.

Background

- Let S be a set with $\#S = n$. A *permutation* of S is a sequence $\pi = \pi_1, \pi_2, \ldots, \pi_n$ obtained by listing the elements of S in some order. We will use $\mathfrak{S}_n$ to denote the set of permutations of $\{1, 2, \ldots, n\}$. This permutation is of length n.
- A *subsequence* is a (not necessarily consecutive) sequence contained within a permutation. We say that $\pi \in \mathfrak{S}_n$ *contains a copy of* $\sigma \in \mathfrak{S}_k$ if there is a subsequence of length k in π with the same relative order as σ. On the other hand, we say that π *avoids* the pattern σ if π does not contain a copy of σ.
  - **Example**: If $\pi = 13254$ contains $\sigma = 132$ as a pattern because the subsequence 154 has the same relative order as σ. We also say that π avoids the pattern 321 since it does not contain a decreasing subsequence of length 3.



We can draw diagrams where the heights of each point is given by the permutation elements. A copy of 132 is bolded.

- We can also consider a permutation avoiding a set of patterns if it does not contain a copy of any pattern in the set. We denote the set of permutations of length n that avoid a set of patterns S as $\text{Av}_n(S)$. This is called the *avoidance set* of S.
  - **Example**: for $\pi \in \mathfrak{S}_n$, $\text{Av}_n(\pi) = \mathfrak{S}_n - \{\pi\}$, since the only permutation with the same length as π that avoids π is π itself.
- Counting the number of permutations avoiding a given pattern (or set of patterns) is a common problem. We call two patterns or sets of patterns *Wilf equivalent* if their avoidance sets have the same cardinalities. This is denoted $\pi \equiv \pi'$.
- A special class of Wilf equivalences are the *trivial Wilf equivalences*, obtained by applying symmetries of the square to the diagram.
  - **Example**: Applying vertical reflection to 132 gives us 231, so we get that $\text{Av}_n(132) = \text{Av}_n(231)$. For patterns in $\mathfrak{S}_3$, applying trivial equivalences like this yield two trivial equivalence classes. Direct counting yields a non-trivial equivalence between the classes. As such, all elements of $\mathfrak{S}_3$ are Wilf equivalent.

Our Problem

- We may also consider *cyclic permutations*, where we let the end of a permutation "wrap around" to the beginning. Accordingly, two permutations are considered the same if one is a rotation of another. We denote cyclic permutations with brackets.
  - **Example**: The only two cyclic permutations of length 3 are $[123]$ and $[132]$, as every other permutation is a rotation of one of these.
- We can consider pattern avoidance in the same way as before.
  - **Example**: As a linear permutation, $\pi = 13254$ avoided 321. If we consider $[\pi]$, however, it no longer does – $[\pi]$ contains $[541]$ (wrapping around) as a copy of $[321]$.
- Since we may choose any rotation we like, we standardize to begin with 1.
- There are only six cyclic permutations of length 4; Callan [1] counted the avoidance sets for these.
- **Our work is on counting the avoidance sets for all *pairs* of length 4 patterns.**

Results

- There are 15 pairs of cyclic permutations of length 4. Applying trivial Wilf equivalences gives us 7 trivial equivalence classes.
  - **Example**: Applying reversal (vertical reflection) to the pair $\{[1234], [1423]\}$ gives us $\{[4321], [3241]\}$, which after rotation to begin with 1 gives us $\{[1432], [1324]\}$. This gives us one such equivalence class.
- We next use counting arguments to determine the size of these classes. A few of the interesting results are:
  - $\#\text{Av}_n([1234], [1243]) = 2(n-2)$ for $n > 2$.
  - $\#\text{Av}_n([1234, [1423]) = 1 + \binom{n-1}{2}$ for all n.
  - $\#\text{Av}_n([1324], [1423]) = 2^{n-2}$ for $n > 1$.

  We demonstrate one such counting argument next, but also note that after counting we have 5 total equivalence classes (direct counting provides two *non-trivial* Wilf equivalences)

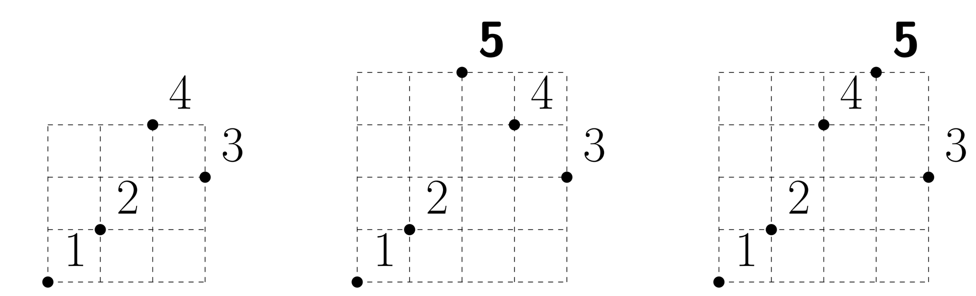
Example Counting Argument

We will now show that $\#\text{Av}_n([1324], [1423]) = 2^{n-2}$ for $n > 1$ as a demonstration of our methods. We proceed by induction. For the base case, we have just one permutation $[12]$, which trivially avoids $[1234], [1432]$. Thus $\#\text{Av}_2 = 1 = 2^{2-2}$. For the inductive step, we will assume that $\#\text{Av}_{n-1} = 2^{n-3}$ and show that every permutation in Av_{n-1} gives us two permutations in Av_n, namely where n is inserted before and after $n-1$, and that no other permutations are possible. This implies that we have twice as many permutations in Av_n than Av_{n-1}, so $\#\text{Av}_n = 2(2^{n-3}) = 2^{n-2}$ as desired.

[1] David Callan. "Pattern Avoidance in Cyclic Permutations". In: (2018).

Example Proof (cont.)

First, we show that every permutation in Av_{n-1} corresponds to two permutations in Av_n by insertion before and after the element $n-1$. Let $[\pi] = [1 \ldots n \ldots]$ be in Av_{n-1}. If we insert n before $n-1$, then the new permutation is of the form $[\pi'] = [1 \ldots (n)(n-1) \ldots]$. Since $[\pi]$ was in Av_{n-1} before insertion, if after insertion it isn't in Av_n, then n must be involved in the offending pattern copies. If $[\pi']$ had a copy of $[1324]$, then if n is involved, it must be the 4 since it is greater than all elements. Since $n-1$ and n are adjacent in π' but not $[1324]$, $n-1$ cannot be the 3. Then the copy replacing n with $n-1$ is in $[\pi]$, contradicting our assumption that $[\pi]$ was in Av_{n-1}. The same line of reasoning shows that π' does not contain $[1423]$, as well as that insertion of n after $n-1$ will still be in Av_n.



Example: A sample permutation in Av_{n-1}, and insertions before and after $n-1$. We see that both of these permutations are in Av_n.

To show that no other permutations are in Av_n, we will show that no other insertion of n will work. This follows from the fact that removing n from a permutation in Av_n must result in a permutation in Av_{n-1}, so every permutation in Av_n must come from inserting n in some permutation in Av_{n-1} Given this, assume that we have $[\pi] \in \text{Av}_{n-1}$, but we insert n before but not adjacent to $n-1$ to get $[\pi'] = [1 \ldots (n) \ldots x \ldots (n-1)]$ in Av_n for some (at least one) x. Then we have a copy of $[1423]$ given by $[1(n)x(n-1)]$, so this is impossible. If we insert n after but not adjacent to n, we have $[1 \ldots (n-1) \ldots x \ldots (n)]$, and we then have a copy of $[1324]$ given by $[1(n-1)x(n)]$, so this is impossible too. So $n-1$ and n must be adjacent, and we have precisely two ways for this to happen. By induction, the proof is complete.

Future Work

- We have also proven results for avoidance sets for triples of cyclic patterns.
- Other work includes examining cyclic shuffle compatibility and generating functions for cyclic permutation statistics.

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