Density and Kappa Value of Integral Sequences with Missing Separations

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Abstract

Let D be a set of positive integers. The density of D is the maximum density of integral sequences in which the separation between any two terms is not in D. The kappa value of D is the parameter involved in the "lonely runner conjecture." The density and kappa value are known for $D = \{1, j, m(j+1)+3\}$ where j is even and $1 \le m \le j-3$. We extend these results to $D = \{1, j, m(j+1)+l\}$ where $l \in \{5, 7\}$, j is even, and $1 \le m \le j-l$.

Background

Let D be a set of positive integers. A sequence S of non-negative integers is called a D-sequence if $|x-y| \notin D$ for any $x, y \in S$. Let $S[n]$ denote $|S \cap \{0, 1, 2, ..., n\}|$. The **density of S** is defined by

$$\delta(S) = \lim_{n\to\infty} \frac{S[n]}{n+1}.$$

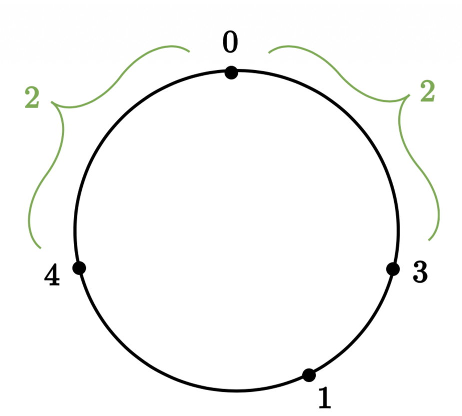
The **density of D** is defined by

$$\boxed{\mu(D) = \sup\{\delta(S) : \text{S is a } D\text{-sequence}\}.} \qquad (1)$$

For example, let $D = \{1, 3, 4\}$. We consider two possible D-sequences.

S_1: ⓪ 1 2 3 4 ⑤ 6 7 8 9 ⑩ 11 12 13 14 ⑮ 16 17 ...

S_2: ⓪ 1 ② 3 4 5 6 ⑦ 8 ⑨ 10 11 12 13 ⑭ 15 ⑯ 17 ...



Let $||x||_s = \min\{x \pmod s, s - x \pmod s\}$. For some $t \in \mathbb{R}$, let $||tD||_s = \min\{||td||_s : d \in D\}$. The **kappa value** of D is defined by

$$\boxed{\kappa(D) = \max\left\{\frac{||tD||_s}{s} : \gcd(s,t) = 1\right\}.} \qquad (2)$$

Known Results

1. For any set D, $\mu(D) \ge \kappa(D)$.
2. Haralambis' Lemma: Let D be a set of positive integers, and let $\alpha \in (0, 1]$. If for every D-sequence S with $0 \in S$ there exists a positive integer n such that $S[n]/(n+1) \le \alpha$, then $\mu(D) \le \alpha$.
3. Let $D = \{1, j, k\}$, where j is even, $k = m(j+1)+3$, and $1 \le m \le j-3$. Then

$$\mu(D) = \kappa(D) = \frac{j(m+1)}{2(j+k)}.$$

Open Problem

What is the density and kappa value of $D = \{1, j, k\}$ where j is even, $k = m(j+1)+l$, and $l \in \{5, 7\}$?

New Result

Let $D = \{1, j, k\}$ where j is even, $k = m(j+1)+l$, $l \in \{5, 7\}$, and $1 \le m \le j-l$. Then

$$\boxed{\mu(D) \ge \kappa(D) \ge \frac{j(m+1)}{2(j+k)}.} \qquad (3)$$

If $l = 5$, then equality holds.

Algorithm to Calculate Kappa Value

Consider $D = \{1, 6, 12\}$. When D contains exactly 3 elements, the denominator of $\kappa(D)$ is the sum of two elements of D. We consider three cases: $d_1 = 7$, $d_2 = 13$, and $d_3 = 18$.

$d_1 = 7$				
t	1	6	12	min
1	1	-1	-2	1
2	2	-2	3	2
3	3	-3	1	1
4	-3	3	-1	1

$d_2 = 13$				
t	1	6	12	min
1	1	6	-1	1
2	2	-1	-2	1
3	3	5	-3	3
4	4	-2	-4	2
5	5	-4	-5	4
6	6	-3	-6	3
7	-6	3	6	3

$d_3 = 18$				
t	1	6	12	min
1	1	6	-6	1
2	2	-6	6	2
3	3	0	0	0
4	4	6	-6	4
5	5	-6	6	5
6	6	0	0	0
7	7	6	6	6
8	8	6	6	6
9	9	0	0	0

We compare $\frac{2}{7}$, $\frac{4}{13}$, and $\frac{6}{18}$ and determine the maximum value to be $\frac{6}{18}$. Thus, $\kappa(D) = \frac{6}{18}$. This method will generate the kappa value for any 3-element D-set.

Kappa Value Data for New Result

We compute the kappa value of $D = \{1, j, k\}$ where j is even and $k = m(j+1)+5$. We show some data for $1 \le m \le 4$.

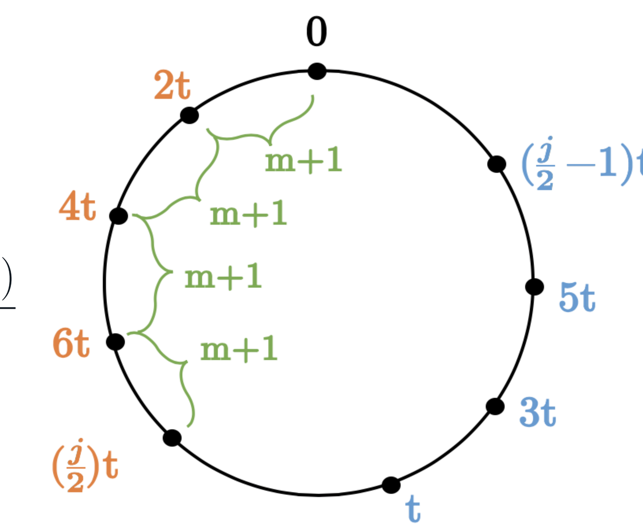
m	j	k	$\kappa(D)$
$m=1$	2	8	1/3
	4	10	4/11
	6	12	6/18
	8	14	8/22
	10	16	10/26
$m=2$	2	11	1/3
	4	15	6/16
	6	19	8/20
	8	23	12/31
	10	27	15/37

m	j	k	$\kappa(D)$
$m=3$	2	14	1/3
	4	20	8/21
	6	26	11/27
	8	32	16/40
	10	38	20/48
$m=4$	2	17	1/3
	4	25	10/26
	6	33	14/31
	8	41	18/42
	10	49	25/59

Proof of New Result

Let $t = \frac{j(m+1)}{2} + (\frac{j}{2} - 1)$. Then

$$t \cdot 1 = \frac{j(m+1)}{2} + (\frac{j}{2} - 1),$$
$$t \cdot j = \frac{j}{2}(j+k) - \frac{j(m+1)}{2},$$
$$t \cdot k = (\frac{mj+j}{2} - 1)(j+k) + \frac{j(m+1)}{2}$$



We conclude that $||tD||_{j+k} = \frac{j(m+1)}{2}$ and

$$\mu(D) \ge \kappa(D) \ge \frac{j(m+1)}{2(j+k)}.$$

If $l = 5$, then we show that $\mu(D) \le j(m+1)/2(j+k)$. Let S be a D-sequence with $0 \in S$. If $S[k] \le mj/2 + 1$, then

$$\frac{S[k]}{k+1} \le \frac{j(m+1)}{2(j+k)}.$$

Therefore, the result follows from Haralambis' Lemma.

If $S[k] = mj/2 + 2$, then we have that this case is impossible. Lastly, if $S[k] = mj/2 + 3$, then let $A_i = S \cap (\{0, ..., j\} + i(j+1))$ for $0 \le i \le m-1$. By assumption, $|A_i| = j/2$ for $0 \le i \le m-1$. Therefore, let $A_i = \{0, 2, 4, ..., e_i - 2, e_i + 1, e_i + 3, ..., j-1\} + i(j+1)$ for $0 \le i \le m-1$ where e_i is even and $e_m \le e_{m-1} \le ... \le e_0 \le j$. For all e_0, we have that $S[j+k-1] \le j(m+1)/2$. The result follows from Haralambis' Lemma. Therefore,

$$\mu(D) = \kappa(D) = \frac{j(m+1)}{2(j+k)}.$$

□

Conjecture

Let $D = \{1, j, k\}$, where j is even, $k = m(j+1)+7$, and $1 \le m \le j-7$. Then

$$\boxed{\mu(D) = \kappa(D) = \frac{j(m+1)}{2(j+k)}.} \qquad (4)$$

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References

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