# A Cyclic Variant of the Erdős - Szekeres Theorem

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## Abstract

We will discuss the Erdős-Szekeres Theorem, which states that every linear permutation of length at least $rs+1$ contains either an increasing subsequence of length $r+1$ or a decreasing subsequence of length $s+1$ (or both). Then, we will give a proof of a cyclic variant of this theorem, which states that every cyclic permutation of length at least $rs+2$ contains either an increasing subsequence of length $r+2$ or a decreasing subsequence of length $s+2$ (or both).

## Background

- Let $S$ be a set with $\#S = n$. A *permutation* of $S$ is a sequence $\pi = \pi_1, \pi_2, \ldots, \pi_n$ obtained by listing the elements of $S$ in some order. This permutation is of length $n$.
  - Ex: $\pi = 13254$ is a permutation of $\{1,2,3,4,5\}$.
- We will use $[n] = \{1,2,\ldots,n\}$ to denote the set of the first $n$ positive integers.
- A *subsequence* is a (not necessarily consecutive) sequence contained within a sequence.
  - Ex: $154$ is a subsequence of $\pi = 13254$.
- An increasing subsequence is a subsequence all the elements are increasing. Similarly, a decreasing subsequence is a subsequence where all the elements are decreasing.
  - Ex: $135$ is an increasing subsequence of $\pi = 13254$.
  - Ex: $54$ is a decreasing subsequence of $\pi = 13254$.

## The Erdős-Szekeres Theorem

**Theorem 1.** *The Erdős-Szekeres Theorem, published in 1935 by Paul Erdős and George Szekeres, states that every permutation with a length at least $rs+1$ must contain an increasing subsequence of length $r+1$ or a decreasing subsequence of length $s+1$ (or both).*[1]

As an example, we see that every permutation of length 9 must contain an increasing subsequence of length $5$ or a decreasing subsequence of length $3$, as $9 = 4 \times 2 + 1$. We can check some of them:

- $172598346$ has $983$ as a subsequence.
- $247596319$ has $24589$ as a subsequence.

But we can find a sequence of length 8 that doesn't have either such subsequence, namely $56781234$.

## A Cyclic Variant

- We may also consider *cyclic permutations*, where we let the end of a permutation "wrap around" to the beginning. In other words, two cyclic permutations are the same if one is a rotation of another. We denote these with brackets.
  - Ex: $[13254] = [32541] = [25413] = [54132] = [41325]$.
- Since we may choose any rotation we like, we standardize to begin with 1.

**Theorem 2.** *Our cyclic variant of the Erdős-Szekeres Theorem states that every cyclic permutation with a length at least $rs+2$ for $0 < r, s$ must contain an increasing subsequence of length $r+2$ or a decreasing subsequence of length $s+2$ (or both).*

As an example, the cyclic permutation $[14837256]$ with length $8 = 3\times2+2$ should contain an increasing subsequence of length $5$, a decreasing subsequence of length $4$. At first glance, it might seem as if it doesn't. But since it is a cyclic sequence, we can find the subsequence $[1876]$, which is equal to $[8761]$.

## Proof

*Proof.* Consider a cyclic permutation $[\pi]$ of $[n]$ with $n = rs+2$. Rotate $[\pi]$ until $n$ is at the end. Consider $\pi'$ on $[n']$ with $n' = n-1 = rs+1$ which is the linear equivalent of this rotation of $[\pi]$, but with $n$ removed.

By the Erdős-Szekeres Theorem, we know there must therefore be an increasing subsequence of length $r+1$ or a decreasing subsequence of length $s+1$ in $\pi'$. Let's call such a subsequence $\pi_{i_1}\pi_{i_2}\pi_{i_3}\ldots\pi_{i_t}$, where $t$ is $r+1$ if the subsequence is increasing or $s+1$ if the subsequence is decreasing.

Now, let's return to $[\pi]$. If $\pi'$ has an increasing subsequence of length $r+1$, keep $[\pi]$ at its current rotation (which is just $\pi'$ followed by $n$). Since $n$ is the largest element in $[\pi]$, this means we have an increasing subsequence of length $r+2$ in $[\pi]$, namely $[\pi_{i_1}\pi_{i_2}\pi_{i_3}\ldots\pi_{i_t}n]$.

Meanwhile, if $\pi'$ has a decreasing subsequence of length $s+1$, we can rotate $[\pi]$ such that $n$ is at the beginning (and is followed by $\pi'$), and once again, since $n$ is the largest element in $[\pi]$, this means we have a decreasing subsequence of length $s+2$ in $[\pi]$, namely $[n\pi_{i_1}\pi_{i_2}\pi_{i_3}\ldots\pi_{i_t}]$.

Therefore, $[\pi]$ has an increasing subsequence of length $r+2$ or a decreasing subsequence of length $s+2$.

This concludes the proof. □

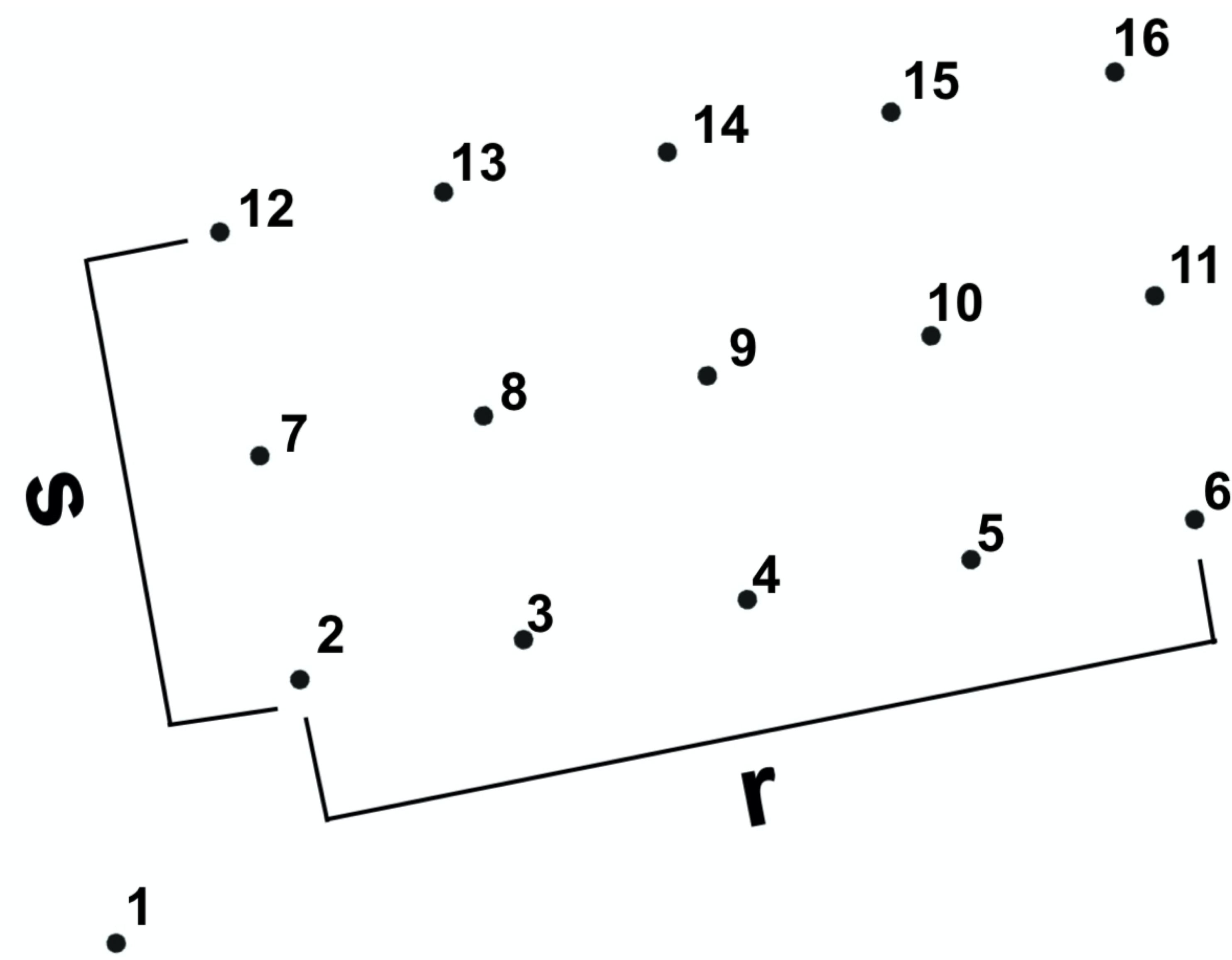
## Remarks

Similarly to the linear variant, the cyclic Erdős-Szekeres Theorem is also a tight bound. In other words, for any $0 < r, s$, we can find a cyclic permutation $[\pi]$ with a length of $rs+1$ such that $[\pi]$ has neither an increasing subsequence of length $r+2$ or a decreasing subsequence of length $s+2$.

Here, we will offer only an example to show that this scales properly.

Though it is difficult to represent $[\pi]$ in a clean closed form, we can draw it as a diagram. In this example, we have $r = 5$ and $s = 3$, and the cyclic permutation $[\pi]$ is $[1, 12, 7, 2, 13, 8, 3, 14, 9, 4, 15, 10, 5, 16, 11, 6]$.

## Remarks (cont.)




The $s$-by-$r$ tilted grid is an example for why the Erdős-Szekeres Theorem is a tight bound, and so it is fairly easy to show that, if we begin an increasing subsequence or end a decreasing subsequence with $1$, we cannot create an increasing subsequence of length $r+2$ or a decreasing subsequence of length $s+2$.

On the other hand, should we ignore 1, then we need only deal with rotations of this tilted grid. Consider each column of the rectangle. Now, we can see that each time we have two successive elements in an increasing subsequence, they must either come from different columns or must loop around from the end of this rotation to the beginning, such as the first and last elements in the increasing subsequence $[723456]$. As we can loop around at most once (because otherwise we would not get a proper subsequence), this means we have that a longest possible increasing subsequence must have a length of $r+1$.

Similarly, looking at rows, we can see that the longest possible decreasing subsequence must have a length of $s+1$.

## Acknowledgements

We would like to thank Dr. Bruce Sagan, Quinn Minnich, Rachel Domagalski, and Jinting Liang, with whom we collaborated to find these results.

[1] P.Erdős and G.Szekeres. "A combinatorial theorem in geometry". In: (1935).