The Period Doubling Route to Chaos

Using the Logistic Equation $X_{next} = rX(1-X)$

This elegantly simple equation, or the results of it, is one of the best known descriptions or definitions of chaos. It was explored by Robert May, a biologist studying population dynamics [1]. What is interesting about it is that when the growth parameter (r) is low the model settles into a steady state. When r is high the steady state breaks apart and chaos ensues; that is, equilibrium becomes impossible. In between are a series of period doubling (bifurcation) transitions when two or more stable states exist.

- Low r = steady population size; or in a fluid, laminar flow.
- Intermediate r = a population fluctuating between several stable population sizes on a regular schedule; in a fluid, a few turbulent eddies form.
- High r = infinitely varying population size, or highly turbulent fluid flow with continuously fluctuating speed and direction.

The equation demonstrates how a deterministic systems (the outcome seems straight forward) can, when pushed, produce non-deterministic and non-linear (chaos) outcomes. At higher values of r (3.5 - 4) the system demonstrates sensitive dependence on initial conditions in that minor changes in the value of r results in markedly different outcomes.

$$X_{next} = rX(1-X)$$

- **X** = population size. For convenience, the "population" is expressed as a fraction between 0 (extinction) and 1 (greatest conceivable population).
- X_{next} is what happens at the next iteration or calculation of the equation.
- **r** = rate of growth, that can be set higher or lower. It is the positive feedback.
- **(1-X)** acts like a regulator (the negative feedback), it keeps the growth within bounds since as x rises, 1-x falls.

The study begins by assigning values to r (say 2.7) and to X (something between 0 and 1) and calculating the equation. The output of this calculation (X_{next}) is used to calculate the equation again (i.e. self referencing). This process is done over and over (iterating the calculation) until something happens to the calculated values, steady state, oscillation, or chaos. The outcomes of the calculations are plotted on a graph so that the paths of the changing values can be easily followed.

A sample calculation of the equation.

$$X = .02 \text{ and } r = 2.7$$
$$X_{next} = rX(1-X)$$
$$X_{next} = (2.7)(.02)(1-.02 = .98)$$
$$X_{next} = .0529$$

The graphs on the next page plot the outcomes of the calculations for a number of values of r, as well as the bifurcation diagram which assembles all the information about all the calculations for all values of r into a single diagram.

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[1] See James Gleick, 1987, Chaos: The Making of a New Science, pages 59-80